3 Bilinear Forms & Euclidean/Hermitian Spaces

Bilinear forms are a natural generalisation of linear forms and appear in many areas of mathematics. Just as linear algebra can be considered as the study of 'degree one' mathematics, bilinear forms arise when we are considering 'degree two' (or quadratic) mathematics. For example, an inner product is an example of a bilinear form and it is through inner products that we define the notion of length in analytic geometry - recall that the length of a vector $\underline{x} \in \mathbb{R}^n$ is defined to be $\sqrt{x_1^2 + ... + x_n^2}$ and that this formula holds as a consequence of Pythagoras' Theorem. In addition, the 'Hessian' matrix that is introduced in multivariable calculus can be considered as defining a bilinear form on tangent spaces and allows us to give well-defined notions of length and angle in tangent spaces to geometric objects. Through considering the properties of this bilinear form we are able to deduce geometric information - for example, the local nature of critical points of a geometric surface.

In this final chapter we will give an introduction to arbitrary bilinear forms on $\mathbb{K}$-vector spaces and then specialise to the case $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. By restricting our attention to thse number fields we can deduce some particularly nice classification theorems. We will also give an introduction to Euclidean spaces: these are $\mathbb{R}$-vector spaces that are equipped with an inner product and for which we can 'do Euclidean geometry', that is, all of the geometric Theorems of Euclid will hold true in any arbitrary Euclidean space. We will discuss the notions of orthogonality (=perpendicularity) and try to understand those linear transformations of a Euclidean space that are length-preserving. We will then generalise to $\mathbb{C}$-vector spaces and consider Hermitian spaces and unitary morphisms - these are the complex analogues of Euclidean spaces, where we make use of the 'conjugation' operation that exists on $\mathbb{C}$.

3.1 Bilinear forms

([1], p.179-182) **Throughout this section $\mathbb{K}$ can be ANY number field. V will always denote a finite dimensional $\mathbb{K}$-vector space.**

In this section we will give the basic definitions of bilinear forms and discuss the basic properties of symmetric and alternating bilinear forms. We will see that matrices are useful in understanding bilinear forms and provide us with a tool with which we can determine properties of a given bilinear form

Definition 3.1.1. Let V be a finite dimensional $\mathbb{K}$-vector space. A $\mathbb{K}$-*bilinear form on* V is a function

$$B : V \times V \to \mathbb{K} \, ; \; (u, v) \mapsto B(u, v),$$

such that

(BF1) for every $u, v, w \in V, \lambda \in \mathbb{K}$ we have

$$B(u + \lambda v, w) = B(u, w) + \lambda B(v, w),$$

(BF2) for every $u, v, w \in V, \lambda \in \mathbb{K}$ we have

$$B(u, v + \lambda w) = B(u, v) + \lambda B(u, w).$$

We say that a $\mathbb{K}$-bilinear form on V, B, is *symmetric* if

$$B(u, v) = B(v, u), \text{ for every } u, v \in V.$$

We say that a $\mathbb{K}$-bilinear form on V, B, is *antisymmetric* if

$$B(u, v) = -B(v, u), \text{ for every } u, v \in V.$$

We denote the set of all $\mathbb{K}$-bilinear forms on V by $\mathrm{Bil}_{\mathbb{K}}(V)$. This is a $\mathbb{K}$-vector space (the $\mathbb{K}$-vector space structure will be discussed in a worksheet/homework).

Remark 3.1.2. 1. The conditions BF1, BF2 that a bilinear form B must satisfy can be restated as saying that

'B is linear in each argument.'

2. We will refer to $\mathbb{K}$-bilinear forms on V as simply 'bilinear forms on V', when there is no confusion on $\mathbb{K}$, or even more simply as 'bilinear forms', when there is no confusion on V.

Example 3.1.3.

1. Let V be a finite dimensional $\mathbb{K}$-vector space and let $\alpha_1, \alpha_2 \in V^* = \text{Hom}_{\mathbb{K}}(V, \mathbb{K})$ be two linear forms. Then,

$$B_{\alpha_1,\alpha_2} : V \times V \to \mathbb{K} \; ; \; (u, v) \mapsto \alpha_1(u)\alpha_2(v),$$

is a bilinear form.[53]

In fact, every bilinear form is a sum of bilinear forms of this type. This requires introducing the notion of **tensor product** which is beyond the scope of this course. You can learn about this in Math 250A, the introductory graduate algebra course.

2. Consider the function

$$B : \mathbb{Q}^3 \times \mathbb{Q}^3 \to \mathbb{Q}, \; ; \; \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right) \mapsto x_1y_1 + 3x_2y_3 - x_3y_1 + 2x_1y_3.$$

Then, it can be checked that B is a $\mathbb{Q}$-bilinear form on $\mathbb{Q}^3$. It is neither symmetric nor antisymmetric.[54]

3. Consider the 'dot product' on $\mathbb{R}^n$

$$\cdot : \mathbb{R}^n \times \mathbb{R}^n \; ; \; (\underline{x}, \underline{y}) \mapsto \underline{x} \cdot \underline{y} = x_1y_1 + ... + x_ny_n.$$

Then, this function is a (symmetric) bilinear form on $\mathbb{R}^n$.

4. The function

$$D : \mathbb{K}^2 \times \mathbb{K}^2 \; ; \; (\underline{x}, \underline{y}) \mapsto \det([\underline{x} \; \underline{y}]),$$

where $[\underline{x} \; \underline{y}]$ is the 2×2 matrix with columns $\underline{x}, \underline{y}$, is an antisymmetric bilinear form on $\mathbb{K}^2$.

5. Let $A \in Mat_n(\mathbb{K})$. Then, we have a bilinear form

$$B_A : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K} \; ; \; (\underline{x}, \underline{y}) \mapsto \underline{x}^t A \underline{y},$$

where $\underline{x}^t = [x_1 \; \cdots \; x_n]$ is the row vector determined by the column vector $\underline{x}$. That B_A is a bilinear form follows from basic matrix arithmetic.

B_A is symmetric if and only if A is symmetric.

B_A is antisymmetric if and only if A is antisymmetric.

It will be shown in homework that,

every bilinear form B on $\mathbb{K}^n$ is of the form $B = B_A$, for some $A \in Mat_n(\mathbb{K})$.

As an example, consider the bilinear form B on $\mathbb{Q}^3$ from Example 2 above. Then, we have

$$B = B_A, \quad \text{where} \quad A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ -1 & 0 & 0 \end{bmatrix}.$$

[53] Check this.

[54] Why?

Indeed, we have

$$[x_1\ x_2\ x_3]\begin{bmatrix}1 & 0 & 2\\ 0 & 0 & 3\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix} = [x_1\ x_2\ x_3]\begin{bmatrix}y_1+2y_3\\ 3y_3\\ -y_1\end{bmatrix} = x_1y_1 + 2x_1y_3 + 3x_2y_3 - x_3y_1.$$

Definition 3.1.4. Let V be a $\mathbb{K}$-vector space, $\mathcal{B} = (b_1, \dots, b_n) \subset V$ an ordered basis and $B \in \mathrm{Bil}_{\mathbb{K}}(V)$. Then, we define *the matrix of B relative to $\mathcal{B}$* to be the matrix

$$[B]_{\mathcal{B}} = [a_{ij}] \in Mat_n(\mathbb{K}), \quad \text{where } a_{ij} = B(b_i, b_j).$$

Moreover, if $B' \in \mathrm{Bil}_{\mathbb{K}}(V)$ is another bilinear form then[55]

$$[B]_{\mathcal{B}} = [B']_{\mathcal{B}} \Leftrightarrow B = B'.$$

Hence, there is a well-defined function

$$[-]_{\mathcal{B}} : \mathrm{Bil}_{\mathbb{K}}(V) \to Mat_n(\mathbb{K})\ ;\ B \mapsto [B]_{\mathcal{B}}.$$

Note that this function is dependent on the choice of $\mathcal{B}$.

Proposition 3.1.5. *Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$, $\mathcal{B} \subset V$ an ordered basis. Then,*

a) $[-]_{\mathcal{B}} : \mathrm{Bil}_{\mathbb{K}}(V) \to Mat_n(\mathbb{K})$ is a bijective $\mathbb{K}$-linear morphism.

b) Let $A \in Mat_n(\mathbb{K})$ and $B_A \in \mathrm{Bil}_{\mathbb{K}}(\mathbb{K}^n)$ be the bilinear form on $\mathbb{K}^n$ defined by A. Then,

$$[B_A]_{\mathcal{S}^{(n)}} = A.$$

c) Let $B \in \mathrm{Bil}_{\mathbb{K}}(\mathbb{K}^n)$ and denote

$$A = [B]_{\mathcal{S}^{(n)}} \in Mat_n(\mathbb{K}).$$

Then, $B_A = B$.

Proof: This is a homework exercise. □

Lemma 3.1.6. *Let V be a $\mathbb{K}$-vector space, $\mathcal{B} = (b_1, \dots, b_n) \subset V$ an ordered basis of V and $B \in \mathrm{Bil}_{\mathbb{K}}(V)$. Then, for any $u, v \in V$ we have*

$$[u]_{\mathcal{B}}^t [B]_{\mathcal{B}} [v]_{\mathcal{B}} = B(u, v) \in \mathbb{K}.$$

Moreover, if $A \in Mat_n(\mathbb{K})$ is such that

$$[u]_{\mathcal{B}}^t A [v]_{\mathcal{B}} = B(u, v),$$

for every $u, v \in V$, then $A = [B]_{\mathcal{B}}$.

Proof: Let $u, v \in V$ and suppose that

$$u = \sum_{i=1}^{n} \lambda_i b_i, \ v = \sum_{j=1}^{n} \mu_j b_j,$$

so that

$$[u]_{\mathcal{B}}^t = [\lambda_1\ \dots\ \lambda_n],\ [v]_{\mathcal{B}} = \begin{bmatrix}\mu_1\\ \vdots\\ \mu_n\end{bmatrix}.$$

[55] Why is this true?

Then, we have

$$\begin{aligned} B(u,v) &= B(\sum_{i=1}^n \lambda_i b_i, \sum_{j=1}^n \mu_j b_j) \\ &= \sum_{i=1}^n \lambda_i B(b_i, \sum_{j=1}^n \mu_j b_j), \quad \text{by BF1,} \\ &= \sum_{i=1}^n \sum_{j=1}^n \lambda_i \mu_j B(b_i, b_j), \quad \text{by BF2.} \end{aligned}$$

Also, we see that

$$[u]_{\mathcal{B}}^t [B]_{\mathcal{B}} [v]_{\mathcal{B}} = [\lambda_1 \ \dots \ \lambda_n][B]_{\mathcal{B}} \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} = \sum_{i=1}^n \sum_{j=1}^n \lambda_i \mu_j B(b_i, b_j).$$ [56]

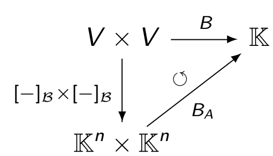
The result follows.

The last statement can be checked by using the fact that

$$x_{ij} = e_i^t X e_j = B(b_i, b_j),$$

for any $X = [x_{ij}] \in Mat_n(\mathbb{K})$. □

Remark 3.1.7. 1. Suppose that V is a $\mathbb{K}$-vector space and $\mathcal{B} = (b_1, \dots, b_n) \subset V$ is an ordered basis. Let $A = [B]_{\mathcal{B}}$ be the matrix of B relative to $\mathcal{B}$. We can interpret Lemma 3.1.6 using the following commutative diagram

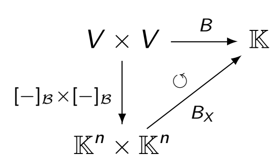


Here we have

$$[-]_{\mathcal{B}} \times [-]_{\mathcal{B}} : V \times V \to \mathbb{K}^n \times \mathbb{K}^n \ ; \ (u, v) \mapsto ([u]_{\mathcal{B}}, [v]_{\mathcal{B}}),$$

and B_A is the bilinear form on $\mathbb{K}^n$ defined by A (Example 3.1.3).

The last statement in Lemma 3.1.6 tells us that if $X \in Mat_n(\mathbb{K})$ is such that we have the commutative diagram



then $X = [B]_{\mathcal{B}}$.

What happens if we choose a different ordered basis $\mathcal{C} \subset V$, **how can we compare $[B]_{\mathcal{B}}$ and $[B]_{\mathcal{C}}$?**

Proposition 3.1.8. *Let V be a $\mathbb{K}$-vector space, $\mathcal{B}, \mathcal{C} \subset V$ ordered bases and $B \in \mathrm{Bil}_{\mathbb{K}}(V)$. Then, if $P = P_{\mathcal{C} \leftarrow \mathcal{B}}$ then*

$$P^t [B]_{\mathcal{C}} P = [B]_{\mathcal{B}},$$

where P^t is the transpose of P.

[56] Check this.

Proof: By Lemma 3.1.6 we know that if we can show that

$$B(u,v) = [u]_{\mathcal{B}}^t P^t [B]_{\mathcal{C}} P [v]_{\mathcal{B}},$$

for every $u, v \in V$, then we must have that

$$[B]_{\mathcal{B}} = P^t [B]_{\mathcal{C}} P.$$

Now, for any $v \in V$ we have that $P[v]_{\mathcal{B}} = [v]_{\mathcal{C}}$, since P is the change of coordinate morphism from $\mathcal{B}$ to $\mathcal{C}$. Thus, for any $u, v \in V$, we have

$$[u]_{\mathcal{B}}^t P^t [B]_{\mathcal{C}} P [v]_{\mathcal{B}} = (P[u]_{\mathcal{B}})^t [B]_{\mathcal{C}} P [v]_{\mathcal{B}} = [u]_{\mathcal{C}}^t [B]_{\mathcal{C}} [v]_{\mathcal{C}} = B(u,v),$$

where we have used that $(XY)^t = Y^t X^t$ and the defining property of $[B]_{\mathcal{C}}$. The result follows. □

3.1.1 Nondegenerate bilinear forms

We will now introduce the important notion of *nondegeneracy* of a bilinear form. Nondegenerate bilinear forms arise throughout mathematics. For example, an inner product is an example of a nondegenerate bilinear form, as is the Lorentzian metric from Einstein's Theory of Special Relativity.

Definition 3.1.9. Let V be a finite dimensional $\mathbb{K}$-vector space, $B \in \mathrm{Bil}_{\mathbb{K}}(V)$. Then, we say that B is *nondegenerate* if the following property holds:

(ND) $\quad B(u,v) = 0, \text{ for every } u \in V \implies v = 0_V.$

If B is not nondegenerate then we say that B is *degenerate*.

Lemma 3.1.10. *Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$, $\mathcal{B} \subset V$ be an ordered basis. Then, B is nondegenerate if and only if $[B]_{\mathcal{B}}$ is an invertible matrix.*

Proof: Suppose that B is nondegenerate. We will show that $A = [B]_{\mathcal{B}}$ is invertible by showing that $\ker T_A = \{\underline{0}\}$. So, suppose that $\underline{x} \in \mathbb{K}^n$ is such that

$$A\underline{x} = \underline{0}.$$

Then, for every $\underline{y} \in \mathbb{K}^n$ we have

$$0 = \underline{y}^t \underline{0} = \underline{y}^t A \underline{x} = B_A(\underline{y}, \underline{x}).$$

As $[-]_{\mathcal{B}} : V \to \mathbb{K}^n$ is an isomorphism we have $\underline{x} = [v]_{\mathcal{B}}$ for some unique $v \in V$. Moreover, if $\underline{y} \in \mathbb{K}^n$ then there is some unique $u \in V$ such that $\underline{y} = [v]_{\mathcal{B}}$. Hence, we have just shown that

$$0 = B_A(\underline{y}, \underline{x}) = [u]_{\mathcal{B}}^t [B]_{\mathcal{B}} [v]_{\mathcal{B}} = B(u,v),$$

by Lemma 3.1.6. Therefore, since B is nondegenerate

$$B(u,v) = 0, \quad \text{for every } u \in V \implies v = 0_V,$$

Hence, $\underline{x} = [v]_{\mathcal{B}} = \underline{0}$ so that $\ker T_A = \{\underline{0}\}$ and A must be invertible.

Conversely, suppose that $A = [B]_{\mathcal{B}}$ is invertible. We want to show that B is nondegenerate so that we must show that if

$$B(u,v) = 0, \quad \text{for every } u \in V,$$

then $v = 0_V$. Suppose that $B(u,v) = 0$, for every $u \in V$. Then, by Lemma 3.1.6, this is the same as

$$0 = B(u,v) = [u]_{\mathcal{B}}^t A [v]_{\mathcal{B}}, \quad \text{for every } u \in V.$$

In particular, if we consider $e_i = [b_i]_{\mathcal{B}}$ then we have

$$0 = e_i^t A [v]_{\mathcal{B}}, \quad \text{for every } i, \implies A[v]_{\mathcal{B}} = \underline{0}.$$

As A is invertible this implies that $[v]_{\mathcal{B}} = \underline{0}$ so that $v = 0_V$, since $[-]_{\mathcal{B}}$ is an isomorphism. □

Corollary 3.1.11. *Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ be a nondegenerate bilinear form. Then,*

$$B(u, v) = 0, \quad \text{for every } v \in V, \implies u = 0_V.$$

BEWARE: this condition is (similar but) different to the one defining nondegeneracy in Definition 3.1.9. Of course, if B is symmetric then this follows from Definition 3.1.9.

Proof: This will be a homework exercise. □

Example 3.1.12.

1. Consider the bilinear form

$$B : \mathbb{Q}^3 \times \mathbb{Q}^3 \to \mathbb{Q} \; ; \; (\underline{x}, \underline{y}) \mapsto x_1 y_2 + x_3 y_2 + x_2 y_1.$$

Then, B is degenerate: indeed, we have

$$A = [B]_{\mathcal{S}^{(n)}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

which is non-invertible.

2. The dot product on $\mathbb{R}^n$ is nondegenerate. This will be shown in a proceeding section.

3. Consider the bilinear form

$$B : Mat_2(\mathbb{Q}) \times Mat_2(\mathbb{Q}) \to \mathbb{Q} \; ; \; (X, Y) \mapsto \mathrm{tr}(XY).$$

Then, B is nondegenerate. Suppose that $X \in Mat_2(\mathbb{Q})$ is such that

$$B(X, Y) = 0, \quad \text{for every } Y \in Mat_2(\mathbb{Q}).$$

Then, in particular, we have

$$B(X, e_{ij}) = 0, \; i, j \in \{1, 2\}.$$

Hence,

$$x_{11} = B(X, e_{11}) = 0, \; x_{12} = B(X, e_{21}) = 0, \; x_{21} = B(X, e_{12}) = 0, \; x_{22} = B(X, e_{22}) = 0,$$

so that $X = 0_2 \in Mat_2(\mathbb{Q})$.

Proposition 3.1.13. *Let V be a $\mathbb{K}$-vector space, $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ a nondegenerate bilinear form. Then, B induces an isomorphism of $\mathbb{K}$-vector spaces*

$$\sigma_B : V \to V^* \; ; \; v \mapsto \sigma_B(v),$$

where

$$\sigma_B(v) : V \to \mathbb{K} \; ; \; u \mapsto \sigma_B(v)(u) = B(u, v).$$

Proof: It is left as an exercise to check that σ_B is well-defined, ie, that σ_B is $\mathbb{K}$-linear and $\sigma_B(v) \in V^*$, for every $v \in V$.

Since we know that $\dim V = \dim V^*$ it suffices to show that σ_B is injective. So, suppose that $v \in \ker \sigma_B$. Then, $\sigma_B(v) = 0 \in V^*$, so that $\sigma_B(v)$ is the zero linear form. Hence, we have $\sigma_B(v)(u) = 0$, for every $u \in V$. Thus, using nondegeneracy of B we have

$$0 = \sigma_B(v)(u) = B(u, v), \quad \text{for every } u \in V, \implies v = 0_V.$$

Hence, σ_B is injective and the result follows. □

Remark 3.1.14. 1. We could have also defined an isomorphism

$$\hat{\sigma}_B : V \to V^*,$$

where

$$\hat{\sigma}_B(v)(u) = B(v, u), \text{ for every } u \in V.$$

If B is symmetric then we have

$$\sigma_B = \hat{\sigma}_B,$$

but this is not the case in general.

2. In fact, Proposition 3.1.13 has a converse: suppose that σ_B induces an isomorphism

$$\sigma_B : V \to V^*.$$

Then, B is nondegenerate. This follows because σ_B is *injective*.[57]

3. Suppose that $\mathcal{B} = (b_1, \dots, b_n) \subset V$ is an ordered basis of V and $\mathcal{B}^* = (b_1^*, \dots, b_n^*) \subset V^*$ is the dual basis (Proposition 1.8.3). **What is the matrix $[\sigma_B]_{\mathcal{B}}^{\mathcal{B}^*}$ of σ_B with respect to $\mathcal{B}$ and $\mathcal{B}^*$?**

By definition we have

$$[\sigma_B]_{\mathcal{B}}^{\mathcal{B}^*} = [[\sigma_B(b_1)]_{\mathcal{B}^*} \cdots [\sigma_B(b_n)]_{\mathcal{B}^*}].$$

Now, for each i, $\sigma_B(b_i) \in V^*$ is a linear form on V so we need to know what it does to elements of V. Suppose that

$$v = \lambda_1 b_1 + \dots + \lambda_n b_n \in V.$$

Then,

$$\sigma_B(b_i)(v) = B(\sum_{k=1}^n \lambda_k b_k, b_i) = \sum_{k=1}^n \lambda_k B(b_k, b_i),$$

and

$$\left(\sum_{j=1}^n B(b_j, b_i) b_j^*\right)(v) = \left(\sum_{j=1}^n B(b_j, b_i) b_j^*\right)\left(\sum_{k=1}^n \lambda_k b_k\right) = \sum_{k=1}^n \lambda_k B(b_k, b_i),$$

so that we must have

$$\sigma_B(b_i) = \sum_{j=1}^n B(b_j, b_i) b_j^*.$$

Hence,

$$\boxed{[\sigma_B]_{\mathcal{B}}^{\mathcal{B}^*} = [B]_{\mathcal{B}}.}$$

It is now clear that B is nondegenerate precisely when the morphism σ_B is an isomorphism.

Definition 3.1.15. Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$. Let $E \subset V$ be a nonempty subset. Then, we define the *(right) B-complement of E in V* to be the set

$$E_r^{\perp} = \{v \in V \mid B(u, v) = 0 \text{ for every } u \in E\};$$

this is a subspace of V.[58]

Similarly, we define the *(left) B-complement of E in V* to be the set

$$E_l^{\perp} = \{v \in V \mid B(v, u) = 0, \text{ for every } u \in E\};$$

[57] Some people actually use this property to *define* nondegeneracy: they say that B is nondegenerate if σ_B is injective. If you think about it, you will see that these two definitions are saying the exact same thing.

[58] Check this.

this is a subspace of V.[59]

If B is (anti-)symmetric then we have that

$$E_l^\perp = E_r^\perp.$$

In this case we write $E^\perp$.

Remark 3.1.16. Let $E \subset V$ be a nonempty subset and $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ be (anti-)symmetric. Then, it is not hard to see that

$$E^\perp = \mathrm{span}_{\mathbb{K}}(E)^\perp.$$

Indeed, we obviously have

$$\mathrm{span}_{\mathbb{K}}(E)^\perp \subset E^\perp,$$

since if $B(u, v) = 0$, for every $u \in \mathrm{span}_{\mathbb{K}}(E)$, then this must also hold for those $u \in E$. Hence, $v \in \mathrm{span}_{\mathbb{K}}(E)^\perp \implies v \in E^\perp$. Conversely, if $v \in E^\perp$, so that $B(e, v) = 0$, for every $e \in E$, then if $w = c_1e_1 + ... + c_ke_k \in \mathrm{span}_{\mathbb{K}}(E)$, then

$$B(w, v) = B(c_1e_1 + ... c_ke_k, w) = c_1B(e_1, v) + ... + c_kB(e_k, w) = 0 + ... + 0 = 0.$$

Proposition 3.1.17. *Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ be (anti-)symmetric and nondegenerate, $U \subset V$ a subspace of V. Then,*

$$\dim U + \dim U^\perp = \dim V.$$

Proof: As B is nondegenerate we can consider the isomorphism

$$\sigma_B : V \to V^*,$$

from Proposition 3.1.13. We are going to show that

$$\sigma_B(U^\perp) = \mathrm{ann}_{V^*}(U) = \{\alpha \in V^* \mid \alpha(u) = 0, \text{ for every } u \in U\}.$$

Indeed, suppose that $w \in U^\perp$. Then, for every $u \in U$, we have

$$\sigma_B(w)(u) = B(u, w) = 0,$$

so that $\sigma_B(w) \in \mathrm{ann}_{V^*}(U)$. Conversely, let $\alpha \in \mathrm{ann}_{V^*}(U)$. Then, $\alpha = \sigma_B(w)$, for some $w \in V$, since σ_B is an isomorphism. Hence, for every $u \in U$, we must have

$$0 = \alpha(u) = \sigma_B(w)(u) = B(u, w),$$

so that $w \in U^\perp$ and $\alpha = \sigma_B(w) \in \sigma_B(U^\perp)$.

Hence, using Proposition 1.8.10, we have

$$\dim U^\perp = \dim \sigma_B(U^\perp) = \dim \mathrm{ann}_{V^*}(U) = \dim V - \dim U.$$

The result follows. □

3.1.2 Adjoints

Suppose that $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ is a nondegenerate symmetric bilinear form on V. Then, we have the isomorphism

$$\sigma_B : V \to V^*,$$

given above.

Consider a linear endomorphism $f \in \mathrm{End}_{\mathbb{K}}(V)$. Then, we have defined the dual of f (Definition 1.8.4)

$$f^* : V^* \to V^* \; ; \; \alpha \mapsto f^*(\alpha) = \alpha \circ f.$$

[59] Check this.

We are going to define a new morphism $f^+ : V \to V$ called the *adjoint of* f: in order to define a morphism we have to define a function and then show that it is linear.

So, given the input $v \in V$ what is the output $f^+(v) \in V$? We have $\sigma_B(v) \in V^*$ is a linear form on V and we define

$$\alpha_v = f^*(\sigma_B(v)) \in V^*.$$

As σ_B is an isomorphism, there must exist a unique $w \in V$ such that $\sigma_B(w) = \alpha_v$. We define $f^+(v) = w$: that is, $f^+(v) \in V$ is the unique vector in V such that

$$\sigma_B(f^+(v)) = f^*(\sigma_B(v)).$$

Hence, for every $u \in V$ we have that

$$\sigma_B(f^+(v))(u) = f^*(\sigma_B(v))(u) \implies \sigma_B(f^+(v))(u) = \sigma_B(v)(f(u)) \implies B(u, f^+(v)) = B(f(u), v).$$

Moreover, since we have

$$f^+ = \sigma_B^{-1} \circ f^* \circ \sigma_B,$$

then we see that f^+ is a linear morphism (it is the composition of linear morphisms, hence must be linear).

Definition 3.1.18. Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ be symmetric and nondegenerate. Suppose that $f \in \mathrm{End}_{\mathbb{K}}(V)$. Then, we define *the adjoint of f (with respect to B)*, denoted f^+, to be the linear morphism

$$f^+ = \sigma_B^{-1} \circ f^* \circ \sigma_B \in \mathrm{End}_{\mathbb{K}}(V).$$

It is the unique endomorphism of V such that

$$B(u, f^+(v)) = B(f(u), v), \quad \text{for every } u, v \in V.$$

We will usually just refer to f^+ as the adjoint of f, the bilinear form B being implicitly assumed known.

Remark 3.1.19. The adjoint of a linear morphism can be quite difficult to understand at first. In particular, given an ordered basis $\mathcal{B} \subset V$, what is $[f^+]_{\mathcal{B}}$?

We use the fact that

$$f^+ = \sigma_B^{-1} \circ f^* \circ \sigma_B,$$

so that

$$\boxed{[f^+]_{\mathcal{B}} = [\sigma_B^{-1} \circ f^* \circ \sigma_B]_{\mathcal{B}} = [\sigma_B^{-1}]^{\mathcal{B}}_{\mathcal{B}^*}[f^*]_{\mathcal{B}^*}[\sigma_B]^{\mathcal{B}^*}_{\mathcal{B}} = [B]_{\mathcal{B}}^{-1}[f]^t_{\mathcal{B}}[B]_{\mathcal{B}}}$$

Hence, if $B = B_A \in \mathrm{Bil}_{\mathbb{K}}(\mathbb{K}^n)$, for some symmetric $A \in \mathrm{GL}_n(\mathbb{K})$, and $f = T_C$, where $C \in Mat_n(\mathbb{K})$, then we have

$$f^+ = T_X, \quad \text{where } X = A^{-1}C^tA.$$

Example 3.1.20. Consider the bilinear form $B = B_A \in \mathrm{Bil}_{\mathbb{Q}}(\mathbb{Q}^3)$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in \mathrm{GL}_3(\mathbb{Q}).$$

Let $f \in \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}^3)$ be the linear morphism

$$f : \mathbb{Q}^3 \to \mathbb{Q}^3 \; ; \; \underline{x} \mapsto C\underline{x},$$

where

$$C = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & 0 \\ -3 & 2 & 5 \end{bmatrix}.$$

Then, the adjoint of f is the morphism

$$f^+ : \mathbb{Q}^3 \to \mathbb{Q}^3 \ ; \ \underline{x} \mapsto \begin{bmatrix} 1 & -3 & -1 \\ 1 & 5 & 0 \\ 0 & 2 & 3 \end{bmatrix} \underline{x}.$$

As a verification, you can check that

$$B\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -3 & -1 \\ 1 & 5 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \right) = B\left(\begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & 0 \\ -3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \right).$$

3.2 Real and complex symmetric bilinear forms

Throughout the remainder of these notes we will assume that $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Throughout this section we will assume that all bilinear forms are symmetric.

When we consider symmetric bilinear forms on real or complex vector spaces we obtain some particularly nice results.[60] For a $\mathbb{C}$-vector space V and symmetric bilinear form $B \in \mathrm{Bil}_{\mathbb{C}}(V)$ we will see that there is a basis $\mathcal{B} \subset V$ such that

$$[B]_{\mathcal{B}} = I_{\dim V}.$$

First we introduce the important *polarisation identity*.

Lemma 3.2.1 (Polarisation identity)**.** *Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ be a symmetric bilinear form. Then, for any $u, v \in V$, we have*

$$B(u, v) = \frac{1}{2}\left(B(u + v, u + v) - B(u, u) - B(v, v)\right).$$

Proof: Left as an exercise for the reader. □

Corollary 3.2.2. *Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ be symmetric and nonzero. Then, there exists some nonzero $v \in V$ such that $B(v, v) \neq 0$.*

Proof: Suppose that the result does not hold: that is, for every $v \in V$ we have $B(v, v) = 0$. Then, using the polarisation identity (Lemma 3.2.1) we have, for every $u, v \in V$,

$$B(u, v) = \frac{1}{2}\left(B(u + v, u + v,) - B(u, u) - B(v, v)\right) = \frac{1}{4}(0 - 0 - 0) = 0.$$

Hence, we must have that $B = 0$ is the zero bilinear form, which contradicts our assumption on B. Hence, ther must exist some $v \in V$ such that $B(v, v) \neq 0$. □

This seemingly simple result has some profound consequences for nondegenerate complex symmetric bilinear forms.

Theorem 3.2.3 (Classification of nondegenerate symmetric bilinear forms over $\mathbb{C}$)**.** *Let $B \in \mathrm{Bil}_{\mathbb{C}}(V)$ be symmetric and nondegenerate. Then, there exists an ordered basis $\mathcal{B} \subset V$ such that*

$$[B]_{\mathcal{B}} = I_{\dim V}.$$

Proof: By Corollary 3.2.2 we know that there exists some nonzero $v_1 \in V$ such that $B(v_1, v_1) \neq 0$ (we know that B is nonzero since it is nondegenerate). Let $E_1 = \mathrm{span}_{\mathbb{C}}\{v_1\}$ and consider $E_1^{\perp} \subset V$.

We have $E_1 \cap E_1^{\perp} = \{0_V\}$: indeed, let $x \in E_1 \cap E_1^{\perp}$. Then, $x = cv_1$, for some $c \in \mathbb{C}$. As $x \in E_1^{\perp}$ we must have

$$0 = B(x, v_1) = B(cv_1, v_1) = cB(v_1, v_1),$$

so that $c = 0$ (as $B(v_1, v_1) \neq 0$). Thus, by Proposition 3.1.17, we must have

$$V = E_1 \oplus E_1^{\perp}.$$

[60] Actually, all results that hold for $\mathbb{C}$-vector space also hold for $\mathbb{K}$-vector spaces, where $\mathbb{K}$ is an algebraically closed field. To say that $\mathbb{K}$ is algebraically closed means that the Fundamental Theorem of Algebra holds for $\mathbb{K}[t]$; equivalently, every polynomial $f \in \mathbb{K}[t]$ can be written as a product of linear factors.