($\Leftarrow$) Conversely, suppose that every vector $v \in V$ can be expressed uniquely as $v = u + w$ for $u \in U$ and $w \in W$. Then, the existence of this expression for each $v \in V$ is simply the statement that $V = U + W$. Moreover, let $x \in U \cap W$, so that $x \in U$ and $x \in W$. Thus, there are $u \in U$ and $w \in W$ (namely, $u = x$ and $w = x$) such that

$$0_V + w = x = u + 0_V,$$

and since U and W are subspaces (so that $0_V \in U, W$) we find, by the uniqueness of an expression for $x \in U \cap W \subset V$, that $u = 0_V = w$. Hence, $x = 0_V$ and $U \cap W = \{0_V\}$. □

1.3 Linear Dependence & $\mathrm{span}_{\mathbb{K}}$

In this section we will make precise the notion of *linear (in)dependence*. This is a fundamental concept in linear algebra and abstracts our intuitive notion of *(in)dependent directions* when we consider the (Euclidean) plane $\mathbb{R}^2$ or (Euclidean) space $\mathbb{R}^3$.

Definition 1.3.1. Let V be a $\mathbb{K}$-vector space[17], and let $\{v_1, \dots, v_n\} \subset V$ be some subset. A *linear relation (over $\mathbb{K}$) among $v_1, \dots, v_n$* is an equation

$$\lambda_1 v_1 + \dots + \lambda_n v_n = 0_V, \tag{1.3.1}$$

where $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ are scalars.

If $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$ then we call (1.3.1) a *trivial linear relation (among $v_1, \dots, v_n$)*.

If at least one of $\lambda_1, \dots, \lambda_n$ is nonzero, so that

$$\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \neq \underline{0}_{\mathbb{K}^n},$$

then we call (1.3.1) a *nontrivial linear relation (among $v_1, \dots, v_n$)*.

Now, let $E \subset V$ be an arbitrary nonempty subset (possibly infinite; NOT necessarily a subspace). We say that E is *linearly dependent (over $\mathbb{K}$)* if there exists $v_1, \dots, v_n \in E$ and a nontrivial linear relation (over $\mathbb{K}$) among $v_1, \dots, v_n$.

If E is not linearly dependent (over $\mathbb{K}$) then we say that E is *linearly independent (over $\mathbb{K}$)*.

Remark 1.3.2. There are some crucial remarks to make:

1. We have defined linear (in)dependence for an arbitrary nonempty subset E of a $\mathbb{K}$-vector space V. In particular, E may be infinite (for example, we could take $E = V$![18]). However, for a subset to be linearly dependent we need only find a linear relation among <u>finitely many</u> vectors in E. Hence, if there is a linear relation (over $\mathbb{K}$) of the form (1.3.1) for some vectors $v_1, \dots, v_n \in V$ and some scalars $\lambda_1, \dots, \lambda_n$ (at least one of which is nonzero), then for *any* subset $S \subset V$ such that $\{v_1, \dots, v_n\} \subset S$, we must have that S is linearly dependent.

2. We will make more precise the notion of linear independence: suppose that $E \subset V$ is a linearly independent set. What does this mean? Definition 1.3.1 defines a subset of V to be linearly independent if it is not linearly dependent. Therefore, a subset E is linearly independent is equivalent to saying that there cannot exist a nontrivial linear relation (over $\mathbb{K}$) among any (finite) subset of vectors in E.

So, in order to show that a subset is linearly independent we need to show that no nontrivial linear relations (over $\mathbb{K}$) can exist among vectors in E. This is equivalent to showing that the only linear relations that exist among vectors in E must necessarily be trivial:

> Suppose that we can write 0_V as a linear combination
> $$\lambda_1 v_1 + \dots + \lambda_n v_n = 0_V,$$
> for some $v_1, \dots, v_n \in E$ and scalars $\lambda_1, \dots, \lambda_n \in \mathbb{K}$. Then, $\lambda_1 = \cdots = \lambda_n = 0$.

[17] Recall our conventions for notation after Remark 1.2.2.
[18] V is always linearly dependent. Why?

Thus, in order to show a given subset E of a vector space E is linearly independent (this could be asked as a homework question, for example) you must show that the above statement is true. This usually requires some thought and ingenuity on your behalf. However, once we have the notion of coordinates (with respect to a basis) we can turn this problem into one involving row-reduction (yay!).

However, to show that a subset $E \subset V$ is linearly dependent you need to find explicit vectors $v_1, \dots, v_n \in E$ and explicit scalars $\lambda_1, \dots, \lambda_n$ (not all of which are zero) so that there is a nontrivial linear relation

$$\lambda_1 v_1 + \dots + \lambda_n v_n = 0_V.$$

This can sometimes be quite difficult! However, once we have the notion of coordinates then we need only try to determine solutions to a matrix equation.

3. We have defined the notion of linear (in)dependence (over $\mathbb{K}$). We will usually omit the phrase 'over $\mathbb{K}$' as it will be assumed implicit that we are seeking linear relations over $\mathbb{K}$ when we are considering subsets of $\mathbb{K}$-vector spaces.

Proposition 1.3.3. *Let V be a $\mathbb{K}$-vector space and $E \subset V$ some nonempty subset. Then, if $0_V \in E$ then E is linearly dependent.*

Proof: We must show that there exists a nontrivial linear relation among some collection of vectors $v_1, \dots, v_n \in E$. We know that $0_V \in V$ and that there is the (obvious?) linear relation

$$1 \cdot 0_V = 0_V,$$

where we have used Proposition 1.2.5. Since we have found a nontrivial linear relation we conclude that E must be linearly dependent. $\square$

Lemma 1.3.4. *Let V be a $\mathbb{K}$-vector space and $E \subset V$ some subset. Then, E is linearly dependent if and only if there exists a vector $v \in E$ that can be written as a linear combination of some of the others.*

Proof: ($\Rightarrow$) Suppose that E is linearly dependent. Then, there exists $v_1, \dots, v_n \in E$ and a nontrivial linear relation

$$\lambda_1 v_1 + \dots + \lambda v_n = 0_V.$$

We may assume, without loss of generality, that $\lambda_1 \neq 0$ and λ_1^{-1} therefore exists. Then, let $v = v_1$ so that we have

$$v = -\lambda_1^{-1}\left(\lambda_2 v_2 + \dots + \lambda_n v_n\right).$$

Hence, $v = v_1$ is a linear combination of some of the other vectors in E.

The converse is left to the reader. $\square$

Corollary 1.3.5. *Let V be a $\mathbb{K}$-vector space, $E \subset V$ a nonempty subset. If E is linearly independent and $v \notin \operatorname{span}_{\mathbb{K}} E$ then $E \cup \{v\}$ is linearly independent.*

Proof: This follows from Lemma 1.3.4: if $E' = E \cup \{v\}$ were linearly dependent then there would exist some $u \in E'$ such that u can be written as a linear combination of other vectors in E'. WE can't have $u = v$, since $v \notin \operatorname{span}_{\mathbb{K}} E$. Hence, $u \in E$ so that it is possible to write u as a linear combination of vectors in E. In this case, E would be linearly dependent by Lemma 1.3.4 which is absurd, since E is assumed linearly independent. Hence, it is not possible for E' to be linearly dependent so it must be linearly independent. $\square$

Question. Why did we care about finding $\lambda_1 \neq 0$? Why did we not just take the nontrivial relation appearing in the proof of Lemma 1.3.4 and move everything to one side except $\lambda_1 v_1$?

Example 1.3.6. Most of the following examples will only concern the linear (in)dependence of finite subsets E. However, I will include a couple of examples where E is infinite to highlight different methods of proof:

1. Consider the $\mathbb{R}$-vector space $\mathbb{R}^3$ and the subset

$$E = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

How do we determine linear (in)dependence of E? We must consider the vector equation

$$\lambda_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = 0_{\mathbb{R}^3}.$$

Then, if there exists a particular $\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \neq 0_{\mathbb{R}^3}$ satisfying this vector equation then E is linearly dependent as we have found a nontrivial linear relation among the vectors in E. Otherwise, E must be linearly independent.

So, determining the linear (in)dependence of $E \subset \mathbb{R}^3$ boils down to **solving the homogeneous matrix equation**

$$A\underline{\lambda} = 0_{\mathbb{R}^3}, \quad \underline{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix},$$

where A is the 3×3 matrix whose columns are the vectors in E. Thus, we must row-reduce A and determine whether there exists a free variable or not: in the language of Math 54, we must determine if there exists a column of A that is not a pivot column.

2. The previous example generalises to any finite subset $E \subset \mathbb{K}^m$, for any $n \in \mathbb{N}$. Let $E = \{v_1, \dots, v_n\} \subset \mathbb{K}^m$ be a subset. Then, determining the linear (in)dependence of E is the same as solving the homogeneous matrix equation

$$A\underline{\lambda} = 0_{\mathbb{K}^m}, \quad \underline{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix},$$

where $A = [v_1 \; v_2 \; \cdots \; v_n]$ is the $m \times n$ matrix whose columns are the vectors in E.

If the only solution to this matrix equation is the zero solution (ie, the only solution is $\underline{\lambda} = 0_{\mathbb{K}^n}$) then E is linearly independent. Otherwise, E is linearly dependent and any nonzero solution you find will determine a nontrivial linear relation among $v_1, v_2, \dots, v_n$.

In general, we want to try and turn a linear (in)dependence problem into one that takes the preceeding form as then we need only row-reduce a matrix and determine pivots.

3. This example is quite subtle and leads to number theoretic considerations: consider the $\mathbb{Q}$-vector space $\mathbb{R}$. Then, the subset $E = \{1, \sqrt{2}\} \subset \mathbb{R}$ is linearly independent (over $\mathbb{Q}$!).

Indeed, consider a linear relation (over $\mathbb{Q}$)

$$a_1.1 + a_2.\sqrt{2} = 0 \in \mathbb{R}, \quad \text{where } a_1, a_2 \in \mathbb{Q}.$$

Assume that E is linearly dependent; we aim to provide a contradiction. Suppose that one of a_1 or a_2 is nonzero (in fact, we must have both of a_1 and a_2 are nonzero. Why?) Then, we have

$$\sqrt{2} = -\frac{a_1}{a_2} \in \mathbb{Q}.$$

However, the Greeks discovered the (heretical[19]) fact that $\sqrt{2}$ is irrational, therefore we can't possibly have that $\sqrt{2} \in \mathbb{Q}$. As such, our intial assumption that E is linearly dependent must be false, so that E is linearly independent (over $\mathbb{Q}$).

If we consider $\mathbb{R}$ as a $\mathbb{R}$-vector space then E is no longer linearly independent: we have

$$-\sqrt{2}.1 + 1.\sqrt{2} = 0 \in \mathbb{R},$$

[19] It is believed that the Pythagorean school in ancient Greece kept the irrationality of $\sqrt{2}$ a secret from the public and that Hippasus was murdered for revealing the secret!

is a nontrivial linear relation (over $\mathbb{R}$) among $1, \sqrt{2}$.

This example highlights the fact that it is important to understand which scalars you are allowed to use in a vector space as properties (for example, linear (in)dependence) can differ when we change scalars.

4. Consider the $\mathbb{R}$-vector space $\mathbb{R}_3[t]$ given in Example 1.2.6. Then, the subset

$$E = \{1, t, t^2, t^3\} \subset \mathbb{R}_3[t],$$

is linearly independent.

We must show that the boxed statement in Remark 1.3.2 holds. So, assume that we have a linear relation

$$\lambda_1.1 + \lambda_2.t + \lambda_3.t^2 + \lambda_4.t^3 = 0_{\mathbb{R}_3[t]},$$

with $\lambda_1, \dots, \lambda_4 \in \mathbb{R}$. Then, by definition, the zero polynomial $0_{\mathbb{R}_4[t]}$ is the polynomial that has all coefficients equal to zero. Therefore, from our remarks in Example 1.2.6 we must have $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ (polynomials are equal if and only if they have equal coefficients).

5. This example might appear to be the same as the previous example but it is actually different: consider $C_{\mathbb{R}}(0,1)$, the $\mathbb{R}$-vector space of continuous functions $f : (0,1) \to \mathbb{R}$. Let $E = \{f_0, f_1, f_2, f_3\}$, where

$$f_i : (0,1) \to \mathbb{R} \ ; \ x \mapsto x^i.$$

Then, E is linearly independent.

Indeed, suppose that we have a linear relation

$$\lambda_0 f_0 + \dots + \lambda_3 f_3 = 0_{C_{\mathbb{R}}(0,1)}, \quad \lambda_1, \dots, \lambda_3 \in \mathbb{R}.$$

Now, this is a linear relation between functions $(0,1) \to \mathbb{R}$, and any two such functions f, g are equal if and only if we have $f(x) = g(x)$, for every $x \in (0,1)$. Hence, we are supposing that

$$\lambda_0 f_0(x) + \lambda_1 f_1(x) + \lambda_2 f_2(x) + \lambda_3 f_3(x) = 0_{C_{\mathbb{R}}(0,1)}(x) = 0, \quad \text{for every } x \in (0,1),$$

$$\implies \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 = 0, \quad \text{for every } x \in (0,1).$$

There are now several ways to proceed: we can either use some calculus or a fundamental fact from algebra. Using calculus, we can differentiate this equation with respect to x repeatedly to obtain that $\lambda_3 = \lambda_2 = \lambda_1 = \lambda_0 = 0$. Alternatively, we can use the following basic fact from algebra: if we assume that one of the λ_i's is nonzero then the polynomial on the LHS of the above equation (considered as a function of x, not a formal expression) can have *at most* three distinct roots. However, since $(0,1)$ is infinite we can choose four distinct roots (for example, $x = 0.1, 0.2, 0.3, 0.4$ are roots), which is absurd. Hence, our assumption that one of the λ_i is nonzero is false, so that $\lambda_0 = \dots = \lambda_3 = 0$ and E is linearly independent.

There is also a linear algebra approach to this problem that will appear on a worksheet.

6. Examples 4 and 5 can be generalised to show that, if $I \subset \mathbb{Z}_{\geq 0}$ is some set of non-negative integers, then the subsets

$$E_1 = \{t^i \mid i \in I\} \subset \mathbb{K}[t], \quad E_2 = \{f_i(x) = x^i \mid i \in I\},$$

are linearly independent.

We now introduce the second fundamental notion concerning vector spaces, that of the linear span of a subset (in [1] this is called the *linear manifold* defined by a subset).

Definition 1.3.7. Let V be a $\mathbb{K}$-vector space, $E \subset V$ some nonempty subset. Then, the $\mathbb{K}$-*linear span of* E is the set of all possible linear combinations of vectors in E,

$$\text{span}_{\mathbb{K}} E = \{\lambda_1 v_1 + \dots + \lambda_n v_n \mid v_1, \dots, v_n \in E, \ \lambda_1, \dots, \lambda_n \in \mathbb{K}\}.$$

If $E \subset V$ is a subset such that $\text{span}_{\mathbb{K}} E = V$, then we say that E *spans* V, or that E *is a spanning set of* V.

Proposition 1.3.8. *Let V be a $\mathbb{K}$-vector space, $E \subset V$ some nonempty subset. Then,* $\text{span}_{\mathbb{K}} E$ *is a vector subspace of V.*

Proof: We will show that $\text{span}_{\mathbb{K}} E$ satsfies Axiom SUB from Definition 1.2.8: let

$$v = \lambda_1 v_1 + ... + \lambda_n v_n, \; u = \mu_1 u_1 + ... + \mu_p u_p \in \text{span}_{\mathbb{K}} E,$$

and $\alpha, \beta \in \mathbb{K}$. Then,

$$\begin{aligned} \alpha u + \beta v &= \alpha(\mu_1 u_1 + ... + \mu_p u_p) + \beta(\lambda_1 v_1 + ... \lambda_n v_n) \\ &= \alpha\mu_1 u_1 + ... + \alpha\mu_p u_p + \beta\lambda_1 v_1 + ... + \beta\lambda_n v_n, \end{aligned}$$

is a linear combination of elements of E. Hence, by the definition of $\text{span}_{\mathbb{K}} E$, $\alpha u + \beta v \in \text{span}_{\mathbb{K}} E$. $\square$

Conversely, we have that every subspace $U \subset V$ is the span of some subset: namely, $\text{span}_{\mathbb{K}} U = U$.

Lemma 1.3.9. *Let V be a $\mathbb{K}$-vector space and $E_1 \subset E_2 \subset V$ nonempty subsets of V. Then,*

$$\text{span}_{\mathbb{K}} E_1 \subset \text{span}_{\mathbb{K}} E_2,$$

and $\text{span}_{\mathbb{K}} E_1$ *is a subspace of* $\text{span}_{\mathbb{K}} E_2$*.*

Proof: Left to the reader. $\square$

Lemma 1.3.10 (Elimination Lemma)**.** *Let V be a $\mathbb{K}$-vector space and $E \subset V$ some nonempty subset. Suppose that E is linearly dependent. Then, there exists a vector $v \in E$ such that, if $E' = E \setminus \{v\}$[20], then*

$$\text{span}_{\mathbb{K}} E = \text{span}_{\mathbb{K}} E'.$$

Hence, we can remove a vector from E without changing the subspace spanned by E.

Proof: Since E is linearly dependent then, by Lemma 1.3.4, there exists a vector $v \in E$ such that v is a linear combination of some other vectors in E, that is

$$v = \lambda_1 v_1 + ... + \lambda_n v_n,$$

with $v_1, ..., v_n \in E$ and $\lambda_1, ..., \lambda_n \in \mathbb{K}$. Moreover, we can assume that $v \neq v_j$, for each j; this is the same as saying that $v \in \text{span}_{\mathbb{K}} E'$. We will show that this v satisfies the conditions of the Lemma.

Now, as $E' \subset E$ then we can use the previous Lemma to conclude that

$$\text{span}_{\mathbb{K}} E' \subset \text{span}_K E.$$

If we can now show that $\text{span}_{\mathbb{K}} E \subset \text{span}_{\mathbb{K}} E'$ then we must have equality

$$\text{span}_{\mathbb{K}} E' = \text{span}_{\mathbb{K}} E.$$

So, let $u \in \text{span}_{\mathbb{K}} E$. Therefore, by the definition of $\text{span}_{\mathbb{K}} E$, we have

$$u = \mu_1 u_1 + ... \mu_k u_k,$$

with $u_1, ..., u_k \in E$ and we can assume that $u_i \neq u_j$ for $i \neq j$. If there is some u_i such that $v = u_i$, then

$$u = \mu_1 u_1 + ... + \mu_{i-1} u_{i-1} + \mu_i v + \mu_{i+1} u_{i+1} + ... + \mu_k u_k.$$

Hence, we have $u_1, ..., u_{i-1}, u_{i+1}, ..., u_k \in E' \subset \text{span}_{\mathbb{K}} E'$ and $v \in \text{span}_{\mathbb{K}} E'$, so that by Proposition 1.3.8, we must have $u \in \text{span}_{\mathbb{K}} E'$.

Now, if $v \neq u_i$, for each i, then each $u_i \in E'$ so that $u \in \text{span}_{\mathbb{K}} E'$. In either case, we have shown that $u \in \text{span}_{\mathbb{K}} E'$ and, since u was arbitrary, we must have $\text{span}_{\mathbb{K}} E \subset \text{span}_{\mathbb{K}} E'$ and the result follows. $\square$

Remark. Lemma 1.3.10 has the following consequence: if E is a finite linearly dependent set that spans the $\mathbb{K}$-vector space V , then there is a subset of E that forms a basis of V. You should already be aware of what a basis is; however, for completeness, we will (re)introduce this notion in an upcoming section in a (perhaps) not so familiar way that fits better with the intuition behind the a basis.

[20]If $S \subset T$ are sets, then define

$$T \setminus S = \{t \in T \mid t \notin S\},$$

the collection of all elements of T that are not elements of S.

References

[1] Shilov, Georgi E., *Linear Algebra*, Dover Publications 1977.