Math 110, Summer 2012 : Bilinear forms review notes

Basics

Definition of a $\mathbb{K}$-bilinear form, (anti-)symmetric property. $\mathbb{K}$-vector space structure of $\mathrm{Bil}_{\mathbb{K}}(V)$. Matrix of a bilinear form with respect to a basis $\mathcal{B}$; fundamental property of this matrix; computing this matrix. Bilinear forms on $\mathrm{Bil}_{\mathbb{K}}(\mathbb{K}^n)$ of the type B_A, $A \in Mat_n(\mathbb{K})$. Nondegeneracy: definition, relation to matrices. B-complements. Definition of adjoint, computing the adjoint of a morphism with respect to a symmetric nondegenerate bilinear form. Examples.

Basic results

Every bilinear form $B \in \mathrm{Bil}_{\mathbb{K}}(\mathbb{K}^n)$ is of the form $B = B_A$. Change of coordiante formula. B nondegenerate if and only if $[B]_{\mathcal{B}}$ is invertible, for any basis $\mathcal{B}$. Nondegenerate bilinear forms induce an isomorphism $\sigma_B : V \to V^*$; the matrix of this isomorphism (with respect to $\mathcal{B}, \mathcal{B}^*$) is $[B]_{\mathcal{B}}$. Dimension formula: $\dim V = \dim U + \dim U^{\perp}$, where $U^{\perp}$ is the B-complement of U. Applying these results to specific examples.

Real/complex symmetric nondegenerate bilinear forms

Polarisation identity. If $B \in \mathrm{Bil}_{\mathbb{C}}(V)$ is symmetric, nondegenerate then there is a basis $\mathcal{B} \subset V$ such that $[B]_{\mathcal{B}} = I_n$. If $B \in \mathrm{Bil}_{\mathbb{R}}(V)$ is symmetric, nondegenerate then there is a basis $\mathcal{B} \subset V$ such that $[B]_{\mathcal{B}}$ is diagonal with nonzero entries being ± 1s (Sylvester's law of inertia). Computing the canonical form of a real symmetric, nondegenerate bilinear form ('completing the square'). Signature of a real symmetric, nondegenerate bilinear form B; it's an invariant of B. Examples.

Euclidean spaces: definitions

Definition of inner product; positive definite property. Definition of Euclidean space, Euclidean morphism, orthogonal transformation. Definition of length (with respect to an inner product), norm function. Orthogonal matrices. Examples of inner (non-) inner products. Projections. Orthogonal sets, orthogonal bases, orthonormal bases. Orthogonal complements.

Euclidean spaces: results

Theorem 3.3.6: Pythagoras' theorem, Cauchy-Schwarz, triangle inequality. $||v|| = 0$ if and only if $v = 0_V$. Inner products are nondegenerate. Euclidean spaces of the same dimension are isomorphic. Computing an isomorphism $(V, \langle , \rangle) \to \mathbb{E}^n$. Euclidean morphisms are injective. Euclidean endmorphisms are isomorphisms. Basic structure of $O(n)$: $T_A \in O(n)$ if and only if $A^t A = I_n$. Euclidean morphisms preserve length. $V = U \oplus U^{\perp}$, for any subspace $U \subset V$. Defining property of $\mathrm{proj}_U v$ as the closest vector in U to v. Orthogonal sets of nonzero vectors are linearly independent. Computing the $\mathcal{B}$-coordinates when $\mathcal{B}$ is an orthogonal basis. Computing $\mathrm{proj}_U v$. Gram-Schmidt process. Examples

Spectral theorem

Normal morphisms/matrices. Orthogonality property of eigenspaces of normal morphisms/matrices. Normal matrices are unitarily diagonalisable (but not necessarily with $\mathbb{R}$-eigenvalues!). Self-adjoint morphisms/matrices: self-adjoint real matrices are symmetric. Self adjoint matrices are orthogonally diagonalisable, with real eigenvalues. Examples