Math 110, Summer 2012: **Exam 1 SOLUTIONS**

1. Let V be a $\mathbb{K}$-vector space, for some number field $\mathbb{K}$. Let $E \subset V$ be a nonempty subset of V.

i) (4 pts) Define what it means for E to be linearly independent (over $\mathbb{K}$). Define what it means for E to be linearly dependent (over $\mathbb{K}$).

ii) (3 pts) Suppose that E is linearly independent and let $F \subset E$ be a nonempty subset. Prove that F is linearly independent.

iii) (5 pts) Suppose that E is linearly dependent. Prove that there exists $v \in E$ such that v can be written as a linear combination

$$v = c_1 v_1 + \dots + c_k v_k, \quad \text{with } c_i \in \mathbb{K}, v_i \in E.$$

iv) (7 pts) Suppose that $E = \{e_1, \dots, e_n\}$ is linearly independent. Let $w \in V$ be such that $w \notin \text{span}_{\mathbb{K}} E$. Prove that $E \cup \{w\}$ is linearly independent.

v) (6 pts) Show that

$$E = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right\} \subset Mat_2(\mathbb{Q}),$$

is linearly independent and extend E to a basis of $Mat_2(\mathbb{Q})$.

Solution:

i) E is linearly independent if, whenever we have a linear relation

$$\lambda_1 v_1 + \dots + \lambda_n v_n = 0_V, \ v_1, \dots, v_n \in E,$$

then $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0 \in \mathbb{K}$. E is linearly dependent if there exists $v_1, \dots, v_m \in E$ and $\lambda_1, \dots, \lambda_m \in \mathbb{K}$, with at least one $\lambda_i \neq 0$, such that

$$\lambda_1 v_1 + \dots + \lambda_m v_m = 0_V.$$

ii) Suppose that we have a linear relation

$$c_1 f_1 + \dots + c_n f_n = 0_V, \ f_1, \dots, f_n \in F.$$

Then, as $f_i \in E$, this is a linear relation among vectors in E. Since E is linearly independent this must be the trivial linear relation. Hence, $c_1 = c_2 = \dots = c_n = 0$, showing that F is linearly independent.

iii) As E is linearly dependent there is a nontrivial linear relation

$$c_1 v_1 + \dots c_n v_n = 0_V, \ v_1, \dots, v_n \in E,$$

where we can assume that $c_1 \neq 0$. Then, we have

$$v_1 = -\frac{1}{c_1}(c_2 v_2 + \dots + c_n v_n),$$

and this is a linear combination of the desired form.

iv) Suppose that we have a linear relation

$$c_1 v_1 + \dots + c_n v_n = 0_V, \; v_1, \dots, v_n \in E \cup \{w\}.$$

If there exists i such that $v_i = w$ and $c_i \neq 0$ then we would have

$$w = -\frac{1}{c_i}(c_1 v_1 + \dots + c_{i-1} v_{i-1} + c_{i+1} v_{i+1} + \dots + c_n v_n) \in \mathrm{span}_{\mathbb{K}} E,$$

which is impossible by our assumption on w. Hence, we must have, for every i, $w \neq v_i$ or, if $v_i = w$, for some i, then $c_i = 0$. Thus, we now have a linear dependence relation among vectors in E which must be the trivial linear relation since E is linearly independent. Hence, the only linear relation that can exist among vectors in $E \cup \{w\}$ is the trivial linear relation.

v) Consider the standard ordered basis $\mathcal{S}$ of $Mat_2(\mathbb{Q})$. Label $E = \{e_1, e_2, e_3\}$, the order being the one written above. Then, we see that

$$[e_1]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, [e_2]_{\mathcal{S}} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, [e_3]_{\mathcal{S}} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

It is easy to see that

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so that the set $\{[e_1]_{\mathcal{S}}, [e_2]_{\mathcal{S}}, [e_3]_{\mathcal{S}}\} \subset \mathbb{Q}^4$ is linearly independent. Hence, since the $\mathcal{S}$-coordinate morphism is an isomorphism the original set E is also linearly independent.

Consider the matrix $e_{11} \in Mat_2(\mathbb{Q})$. Then,

$$[e_{11}]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \sim I_4,$$

so that

$$\mathrm{span}_{\mathbb{Q}}\{[e_1]_{\mathcal{S}}, [e_2]_{\mathcal{S}}, [e_3]_{\mathcal{S}}, [e_{11}]_{\mathcal{S}}\} = \mathbb{Q}^4.$$

Hence, as the $\mathcal{S}$-coordinate morphism is an isomorphism we must have

$$\mathrm{span}_{\mathbb{Q}}\{e_1, e_2, e_3, e_{11}\} = Mat_2(\mathbb{Q}),$$

and $E \cup \{e_{11}\}$ is a basis of $Mat_2(\mathbb{Q})$.

2. i) (3 pts) Let $\mathcal{B} = (b_1, \dots, b_n) \subset V$ be an ordered subset of the $\mathbb{K}$-vector space V. Define what it means for $\mathcal{B}$ to be an ordered basis of V (*You can use ANY definition here.*)

ii) (2 pts) Suppose that $E \subset V$ is a linearly independent subset of a finite dimensional $\mathbb{K}$-vector space V. What is the allowed possible size of E?

iii) (6 pts) Suppose that V is a $\mathbb{K}$-vector space such that $\dim_{\mathbb{K}} V = n$. Let $E \subset V$ be a linearly independent subset of size $|E| = n$. Prove that $\mathrm{span}_{\mathbb{K}} E = V$. (*Hint: Use a 'proof by contradiction' argument and try to contradict your answer for ii) above.*)

iv) (6 pts) Consider the ordered subset

$$\mathcal{B} = (f_1, f_2, f_3) \subset \mathbb{Q}^{\{1,2,3\}} = \{f : \{1,2,3\} \to \mathbb{Q}\},$$

where

$$f_1(1) = 1, f_1(2) = 0, f_1(3) = -1, \; f_2(1) = 1, f_2(2) = 0, f_2(3) = 1, \; f_3(1) = 0, f_3(2) = 1, f_3(3) = 1.$$

Prove that $\mathcal{B}$ is linearly independent. Deduce that $\mathcal{B}$ is a basis of $\mathbb{Q}^{\{1,2,3\}}$.

v) (5 pts) Let $\mathcal{S} = (e_1, e_2, e_3) \subset \mathbb{Q}^{\{1,2,3\}}$ be the standard ordered basis of $\mathbb{Q}^{\{1,2,3\}}$. Determine the change of coordinate matrix $P_{\mathcal{B}\leftarrow\mathcal{S}}$.

vi) (3 pts) Suppose that $f \in \mathbb{Q}^{\{1,2,3\}}$ is such that

$$[f]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

Is $f \in \text{span}_{\mathbb{Q}}\{f_1, f_3\}$? Justify your answer.

Solution:

i) $\mathcal{B}$ is a basis of V if $\mathcal{B}$ is linearly independent and $\text{span}_{\mathbb{K}}\mathcal{B} = V$. It is an ordered basis whenever it is also an ordered set.

ii) Let $n = \dim V$. Then, we must have $|E| \leq n$.

iii) Suppose that $\text{span}_{\mathbb{K}}E \neq V$. Then, there is some $v \in V$ such that $v \notin \text{span}_{\mathbb{K}}E$. Hence, by a result from class, the set

$$E' = E \cup \{v\},$$

is linearly independent and has size $|E'| = n + 1$. However, this contradicts ii). Therefore, our initial assumption that $\text{span}_{\mathbb{K}}E \neq V$ must be false, so that $\text{span}_{\mathbb{K}}E = V$.

iv) Consider the standard ordered basis $\mathcal{S} = (e_1, e_2, e_3) \subset \mathbb{Q}^{\{1,2,3\}}$. Then,

$$[f_1]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \; [f_2]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \; [f_3]_{\mathcal{S}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

We see that

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \sim I_3,$$

so that $\{[f_1]_{\mathcal{S}}, [f_2]_{\mathcal{S}}, [f_3]_{\mathcal{S}}\}$ is linearly independent. Hence, as the $\mathcal{S}$-coordinate morphism is an isomorphism, we obtain that $\mathcal{B}$ is linearly independent. Since $\mathbb{Q}^{\{1,2,3\}}$ is 3 dimensional and $\mathcal{B}$ is a linearly independent set with 3 elements, it must be a basis.

v) We have

$$P_{\mathcal{B}\leftarrow\mathcal{S}} = P_{\mathcal{S}\leftarrow\mathcal{B}}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}.$$

vi) No: we must have

$$f = f_1 - 2f_2.$$

If $f \in \text{span}_{\mathbb{Q}}\{f_1, f_3\}$ then there exists $a, b \in \mathbb{Q}$ such that

$$f_1 - 2f_2 = f = af_1 + bf_3 \implies (1-a)f_1 - 2f_2 - bf_3 = 0_{\mathbb{Q}^{\{1,2,3\}}},$$

which is impossible since $\mathcal{B}$ is linearly independent.

3. i) (6 pts) Define the image im$f$ of a linear morphism $f : V \to W$ and the rank of f, rank$f$. Define the rank of an $m \times n$ matrix $A \in Mat_{m,n}(\mathbb{K})$, rank$A$.

ii) (7 pts) Prove: if rank$f = \dim V$ then f is surjective.

iii) (5 pts) Prove: if $A \in Mat_{m,n}(\mathbb{K}), B \in Mat_{n,p}(\mathbb{K})$ and rank$A = r$, rank$B = s$, then rank$AB \leq r$.

iv) (7 pts) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Determine $r = \text{rank}A$ and find $P, Q \in \text{GL}_3(\mathbb{Q})$ such that

$$Q^{-1}AP = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Solution:

i) The image of f is the set

$$imf = \{w \in W \mid \exists v \in V \text{ such that } f(v) = w\} = \{f(v) \mid v \in V\},$$

and the rank of f is rank$f = \dim \text{im} f$.

The rank of an $m \times n$ matrix is rank$A = \dim \text{im}\, T_A$, where $T_A : \mathbb{K}^n \to \mathbb{K}^m$ is the morphism defined by A.

ii) Suppose that rank$f = \dim W$. Thus, $\dim \text{im} f = \dim W$. Hence, since im$f \subset W$ is a subspace and has the same dimension as W we must have im$f = W$. Therefore, for every $w \in W$, there is some $v \in V$ such that $f(v) = w$. This implies that f is surjective.

iii) Consider the morphisms $T_A \in \text{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^m)$, $T_B \in \text{Hom}_{\mathbb{K}}(\mathbb{K}^p, \mathbb{K}^n)$. Then, we have rank$AB = \dim \text{im}\, T_{AB}$. Now, we have

$$\text{im}\, T_{AB} = \{AB\underline{x} \mid \underline{x} \in \mathbb{K}^p\},$$

and if $\underline{y} = T_{AB}(\underline{x}) \in \text{im}\, T_{AB}$ then,

$$\underline{y} = AB\underline{x} = A(B\underline{x}) = A\underline{z} \in \text{im}\, T_A.$$

Hence, $\text{im}\, T_{AB} \subset \text{im}\, T_A$ is a subspace. Therefore, we must have

$$\text{rank}AB = \dim \text{im}\, T_{AB} \leq \dim \text{im}\, T_A = \text{rank}A = r.$$

iv) Since

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim I_3,$$

we must have that rank$A = 3$ (there is a pivot in every column so that T_A is surjective. Hence, $\dim \text{im}\, T_A = 3$). As T_A is a surjective morphism between spaces of the same dimension it must be an isomorphism. Hence, A is invertible so that we can take

$$Q = I_3,\ P = A^{-1} = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & -1/4 & 3/4 \\ -1/2 & 1/4 & 1/4 \end{bmatrix}.$$

4. i) (4 pts) Let $f \in \mathrm{End}_{\mathbb{C}}(V)$, with V a finite dimensional $\mathbb{C}$-vector space. Define what it means for $\lambda \in \mathbb{C}$ to be an eigenvalue of f. Define the geometric and algebraic multiplicity of λ.

ii) (4 pts) Let $f \in \mathrm{End}_{\mathbb{C}}(V)$, with V a finite dimensional $\mathbb{C}$-vector space. Define what it means for f to be diagonalisable. Give a criterion for f to be diagonalisable using the notions of geometric and algebraic multiplicity of eigenvalues.

iii) (7 pts) Let $f \in \mathrm{End}_{\mathbb{C}}(V)$, where $\dim V = 7$. Suppose that f is non-surjective, diagonalisable and such that $\dim \mathrm{im} f = 1$. Prove that f admits exactly one nonzero eigenvalue λ and that $E_\lambda = \mathrm{im} f$, where E_λ is the λ-eigenspace.

Consider the endomorphism

$$f : Mat_2(\mathbb{C}) \to Mat_2(\mathbb{C}) \ ; \ A \mapsto A + A^t,$$

where A^t is the transpose of A.

iv) (4 pts) Determine the eigenvalues of f and their algebraic multiplicities.

v) (6 pts) Prove that f is diagonalisable and find a basis $\mathcal{B} \subset Mat_2(\mathbb{C})$ such that $[f]_{\mathcal{B}}$ is diagonal.

For iv)-v) you may want to use the standard ordered basis

$$\mathcal{S} = (e_{11}, e_{12}, e_{21}, e_{22}) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \subset Mat_2(\mathbb{C}).$$

Solution:

i) λ is an eigenvalue of f if there is some nonzero $v \in V$ such that $f(v) = \lambda v$.

The geometric multiplicity of λ is $\dim \ker(f - \lambda \mathrm{id}_V)$. If $\chi_f(t)$ is the characteristic polynomial of f then

$$\chi_f(t) = (t - \lambda)^{n_\lambda} g,$$

where $g \in \mathbb{C}[t]$ is such that $g(\lambda) \neq 0$. Then, the algebraic multiplicity of λ is n_λ.

ii) f is diagonalisable if there exists a basis $\mathcal{B} \subset V$ consisting of eigenvectors of f.

f is diagonalisable if and only if, for every eigenvalue of f, the algebraic and geometric multiplicities are equal.

iii) As f is non-surjective then then it must also be non-injective so that $\ker f \neq \{0_V\}$. Hence, we must have that 0 is an eigenvalue of f. Using $\dim \mathrm{im} f = 1$, the Rank Theorem implies that $\dim \ker f = 7 - 1 = 6$, so that the geometric multiplicity of 0 is 6. Since f is assumed diagonalisable we must also have the algebraic multiplicity of 0 is 6. Hence, the characteristic polynomial of f is of the form

$$\chi_f(t) = t^6(t - a),$$

because $\deg \chi_f = 7$, and where $a \neq 0$. Thus, a is a nonzero eigenvalue of f and is the only such.

Moreover, let v be an eigenvector with associated eigenvalue a. Then, $f(v) = av$, so that $v \in \mathrm{im} f$. Conversely, suppose that $w = f(v)$. Then, as f is diagonalisable, we have

$$V = E_0 \oplus E_a = \ker f \oplus E_a.$$

Hence, $v = z + u$, where $z \in \ker f$, $u \in E_a$. So, $w = f(v) = f(z+u) = f(z) + f(u) = 0_V + f(u) = f(u) = au$. Hence, as E_a is a subspace, we have $w \in E_a$ so that $\mathrm{im} f = E_a$.

iv) Consider the standard ordered basis $\mathcal{S}$ of $Mat_2(\mathbb{C})$ given below. Then,

$$B \stackrel{def}{=} [f]_{\mathcal{S}} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Hence, we have

$$\chi_f(t) \det(B - tI_4) = t(t-2)^3,$$

so that the eigenvalues of f are 0 and 2 with algebraic multiplicities 1 and 3 (respectively).

v) We have

$$B - 2I_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so that

$$E_2^B = \text{span}_{\mathbb{C}}\{e_1, e_4, e_2 + e_3\} \implies E_2^f = \{e_{11}, e_{22}, e_{12} + e_{21}\}.$$

Similarly,

$$E_0^B = \text{span}_{\mathbb{C}}\{e_2 - e_3\} \implies E_0^f = \{e_{12} - e_{21}\}.$$

Hence, since the geometric multiplicity of 2 is 3 and the geometric multiplicity of 0 is 1, we must have that f is diagonalisable. Moreover, if we let

$$\mathcal{B} = (e_{11}, e_{22}, e_{12} + e_{21}, e_{12} - e_{21}),$$

then

$$[f]_{\mathcal{B}} = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 0 \end{bmatrix}.$$

5. Consider the following endomorphism

$$L_A : Mat_n(\mathbb{C}) \to Mat_n(\mathbb{C}) \; ; \; B \mapsto AB, \quad \text{where } A \in Mat_n(\mathbb{C}).$$

i) (4 pts) Define what it means for L_A to be nilpotent. Define what it means for A to be nilpotent.

ii) (2 pts) Define the exponent of L_A, $\eta(L_A)$. Define the exponent of A, $\eta(A)$.

iii) (4 pts) Prove: A is nilpotent if and only if L_A is nilpotent.

Now suppose that $n = 2$ and $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

iv) (2 pts) Using iii) deduce that L_A is nilpotent. What is $\eta(L_A)$?

v) (3 pts) Let $\mathcal{S} = (e_{11}, e_{12}, e_{21}, e_{22})$ be the standard ordered basis of $Mat_2(\mathbb{C})$. Determine $X = [L_A]_{\mathcal{S}}$.

vi) (7 pts) Determine an ordered basis $\mathcal{B} = (b_1, b_2, b_3, b_4) \subset Mat_2(\mathbb{C})$ such that $[L_A]_{\mathcal{B}}$ is a block diagonal matrix, each block being a 0-Jordan block.

vii) (3 pts) Determine the partition associated to X, $\pi(X)$. Is X similar to the following matrix

$$Y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}?$$

Justify your answer.

Solution:

i) L_A is nilpotent if there is some $r \in \mathbb{N}$ such that $L_A^r = 0 \in \mathrm{End}_{\mathbb{C}}(Mat_n(\mathbb{C}))$. A is nilpotent if there is some $r \in \mathbb{N}$ such that $A^r = 0 \in Mat_n(\mathbb{C})$.

ii) $\eta(L_A)$ is the smallest $r \in \mathbb{N}$ such that $L_A^r = 0$ while $L_A^r \neq 0$. $\eta(A)$ is the smallest $r \in \mathbb{N}$ such that $A^r = 0$ while $A^{r-1} \neq 0$.

iii) Suppose that A is nilpotent. Then, there is some $r \in \mathbb{N}$ such that $A^r = 0 \in Mat_n(\mathbb{C})$. Hence, for any $B \in Mat_n(\mathbb{C})$ we have

$$L_A^r(B) = A^r B = 0.B = 0 \in Mat_n(\mathbb{C}),$$

so that $L_A^r = 0 \in \mathrm{End}_{\mathbb{C}}(Mat_n(\mathbb{C}))$ and L_A is nilpotent. Conversely, if $L_A^r = 0$ then we must have

$$0 = L_A^r(I_n) = A^r I_n = A^r,$$

so that A is nilpotent.

iv) As A is nilpotent ($A^2 = 0$) we have that L_A is nilpotent. Since $L_A \neq 0$ while $L_A^2 = 0$ (consider the proof of iii)) then $\eta(L_A) = 2$.

v) We have

$$L_A(e_{11}) = e_{11} + e_{21},\ L_A(e_{12}) = e_{12} + e_{22},\ L_A(e_{21}) = -e_{11} - e_{21},\ L_A(e_{22}) = -e_{12} - e_{22},$$

so that

$$X = [L_A]_{\mathcal{S}} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

vi) We follow the algorithm from the notes for the matrix X, making sure to translate our answer back to $Mat_2(\mathbb{C})$ at the end.

We have

$$H_2 = \ker T_{X^2} = Mat_2(\mathbb{C}),\ H_1 = \ker T_X = \{\underline{x} \in \mathbb{C}^4 \mid x_1 - x_3 = 0,\ x_2 - x_4 = 0\} = \mathrm{span}_{\mathbb{C}} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \right\}.$$

Then, we have

$$H_2 = H_1 \oplus G_2,$$

and we can take

$$G_2 = \mathrm{span}_{\mathbb{C}}\{e_1, e_2\}.$$

Set $S_1 = \{Xe_1, Xe_2\} = \{e_1 + e_3, e_2 + e_4\}$. Then, we have

$$H_1 = H_0 \oplus \mathrm{span}_{\mathbb{C}} S_1 \oplus G_1.$$

However, since $H_0 = \{0\}$ and S_1 is linearly independent (so that $\dim \mathrm{span}_{\mathbb{C}} S_1 = 2$) we have $G_1 = 0$.

Therefore, the table we obtain looks like

$$\begin{matrix} e_1 & e_2 \\ Xe_1 & Xe_2 \end{matrix},$$

so if we set $\mathcal{B} = (Xe_1, e_1, Xe_2, e_2)$ then

$$[L_A]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

vii) The partition associated to X is

$$\pi(X) : 2^2 \leftrightarrow 2 + 2 = 4.$$

Therefore, X is not similar to Y since

$$\pi(Y) : 1^2 2 \leftrightarrow 1 + 1 + 2 = 4,$$

so that $\pi(X) \neq \pi(Y)$. A result from class states that two nilpotent matrices are similar if and only if they have the same associated partitions.