Worksheet 6/27. Math 110, Summer 2012

*An asterisk * denotes a harder problem. Speak to your neighbours, these problems should be discussed.*

Finding bases

0. This problem shows you how to find vectors $v \notin \text{span}_{\mathbb{K}} E$, for non-spanning subsets $E \subset V$, when V admits a 'standard basis'.

Consider $V = \mathbb{K}^n$. Then, we have the standard ordered basis $\mathcal{S}^{(n)} = (e_1, \dots, e_n)$. Let $E = \{a_1, \dots, a_k\} \subset V$ is a non-spanning subset. Consider the vector of variables

$$\underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Form the matrix equation

$$A\underline{x} = \underline{b},$$

where $A = [a_1 \cdots a_k]$. Then, row-reducing the augmented matrix $[A\ \underline{b}]$ will give some $n \times (k+1)$ matrix

$$[U\ \underline{c}] = \begin{bmatrix} U' & \underline{c}' \\ 0 & p_1(\underline{b}) \\ \vdots & \vdots \\ 0 & p_r(\underline{b}) \end{bmatrix},$$

where U' is the nonzero part of U and $p_i(\underline{b})$ are linear expressions in the variables $b_1, \dots, b_n$. Then, $\underline{y} \in \text{span}_{\mathbb{K}} E$ if and only if $p_1(\underline{y}) = \dots = p_r(\underline{y}) = 0$. (You DO NOT have to show this. If you have time think about why this is true.)

For example, consider the subset

$$E = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} \subset \mathbb{Q}^3,$$

and row-reduce

$$\begin{bmatrix} 1 & 1 & b_1 \\ 0 & -1 & b_2 \\ -1 & 1 & b_3 \end{bmatrix} \sim ?$$

What are the linear equations defining $\text{span}_{\mathbb{K}} E$ in this case?

Can you see how to generalise this to the case $V = \mathbb{K}^S$, for a finite set S? In particular, you can now determine the equations defining $\text{span}_{\mathbb{K}} E$, for $E \subset Mat_{m,n}(\mathbb{K})$.

1. The following subsets $E \subset V$ are NOT bases of the $\mathbb{K}$-vector space V. Explain why.

In each case obtain a basis of V using E: so, you will either have to make E smaller or larger to obtain a basis.

$$E = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\} \subset \mathbb{Q}^3 = V,$$

$$E = \left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\} \subset \mathbb{R}^2 = V,$$

$$E = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \subset \mathbb{Q}^3 = V,$$

$$E = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \subset \mathbb{Q}^4 = V,$$

$$E = \left\{ e_{12}, e_{22}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \right\} \subset Mat_2(\mathbb{C}) = V.$$

(*If E is not a spanning set, you will find it useful to use the first part of this worksheet to help you out.*)