Math 110, Summer 2012 Short Homework 9 (SOME) SOLUTIONS

Due Thursday 7/26, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

0. Was this homework assignment too easy/too difficult/about right? Any other comments are welcome.

Calculations

1. Give an example of a nondegenerate antisymmetric bilinear form on $\mathbb{Q}^4$.

Solution: The bilinear form $B_A \in \mathrm{Bil}_{\mathbb{Q}}(\mathbb{Q}^4)$, where

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

satisfies the required properties as A is antisymmetric and invertible.

2. Determine the matrix of B with respect to the given basis $\mathcal{B}$. State whether the bilinear form is symmetric/antisymmetric/neither and if it is nondegenerate:

i) $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}\ ;\ (\underline{x}, \underline{y}) \mapsto x_1y_1 + 3x_2y_2 + y_3x_2 - 10x_3y_2,\ \mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right).$

ii) $B : Mat_3(\mathbb{Q}) \times Mat_3(\mathbb{Q}) \to \mathbb{Q}\ ;\ (X, Y) \mapsto \mathrm{tr}(XY),$

$$\mathcal{B} = (e_{11}, e_{12} - e_{21}, e_{32}, e_{13} - e_{31}, e_{13} + e_{31}, e_{22} + e_{33}, e_{33}, e_{23} - 2e_{32}, e_{12} + e_{11}).$$

Solution:

i) We have $[B]_{\mathcal{B}} = P^t[B]_{\mathcal{S}^{(3)}}P$, where

$$[B]_{\mathcal{S}^{(3)}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -10 & 0 \end{bmatrix},\ P = P_{\mathcal{S}^{(3)} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Hence,

$$[B]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 11 \\ 1 & 1 & -9 \\ 0 & 2 & 4 \end{bmatrix}.$$

B is neither symmetric nor antisymmetric as the above matrix is neither. Since $[B]_{\mathcal{B}}$ is invertible then B is nondegenerate.

ii) We have $[B]_{\mathcal{B}} = P^t[B]_{\mathcal{S}^{(3,3)}}P$, where

$$\mathcal{S}^{(3,3)} = (e_{11}, e_{12}, e_{13}, e_{21}, e_{22}, e_{23}, e_{31}, e_{32}, e_{33}),$$

and

$$P = P_{\mathcal{S}^{(3)} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$[B]_{\mathcal{S}^{(3)}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then,

$$[B]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -4 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the matrix $[B]_{\mathcal{B}}$ is symmetric and invertible, the bilinear form B is symmetric and nondegenerate.

3. Determine the adjoint of f with respect to B:

i) $B : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$; $(\underline{x}, \underline{y}) \mapsto \underline{x} \cdot \underline{y}$, $\quad f : \mathbb{R}^4 \to \mathbb{R}^4$; $\underline{x} \mapsto A\underline{x}$, where

$$A = \begin{bmatrix} \pi & -1 & 0 & 0 \\ e^2 & \sqrt{2} & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -\sqrt{5} & 0 & 10 & 1 \end{bmatrix}.$$

ii) $B : Mat_2(\mathbb{Q}) \times Mat_2(\mathbb{Q}) \to \mathbb{Q}$; $(X, Y) \mapsto \mathrm{tr}(XY)$, $\quad f : Mat_2(\mathbb{Q}) \to Mat_2(\mathbb{Q})$; $X \mapsto X^t$.

Proofs

4. Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ and $\mathcal{B} \subset V$ be an ordered basis. Prove that $[-]_{\mathcal{B}} : \mathrm{Bil}_{\mathbb{K}}(V) \to Mat_n(\mathbb{K})$ ($n = \dim V$) is linear and bijective.

Solution: Let $B, B' \in \mathrm{Bil}_{\mathbb{K}}(V)$, $\lambda, \mu \in \mathbb{K}$. Then,

$$[\lambda B + \mu B']_{\mathcal{B}} = [a_{ij}], \quad a_{ij} = \lambda B(b_i, b_j) + \mu B'(b_i, b_j).$$

Also,

$$\lambda[B]_{\mathcal{B}} + \mu[B]_{\mathcal{B}} = \lambda[c_{ij}] + \mu[d_{ij}] = [e_{ij}], \quad c_{ij} = B(b_i, b_j), d_{ij} = B'(b_i, b_j) \implies e_{ij} = \lambda B(b_i, b_j) + \mu B'(b_i, b_j).$$

Hence, for every i, j, we have $e_{ij} = a_{ij}$ so that

$$[\lambda B + \mu B']_{\mathcal{B}} = \lambda[B]_{\mathcal{B}} + \mu[B']_{\mathcal{B}},$$

and $[-]_{\mathcal{B}}$ is linear.

Suppose that $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ is such that $[B]_{\mathcal{B}} = 0_n$. Then, for any $u, v \in V$, we have

$$B(u, v) = [u]_{\mathcal{B}}^t [B]_{\mathcal{B}} [v]_{\mathcal{B}} = [u]_{\mathcal{B}}^t 0_n [v]_{\mathcal{B}} = 0,$$

so that $B = 0$ is the zero bilinear form. Hence, $[-]_{\mathcal{B}}$ is injective.

Suppose that $X \in Mat_n(\mathbb{K})$. Consider the function

$$B : V \times V \to \mathbb{K} \; ; \; (u, v) \mapsto [u]_{\mathcal{B}}^t X [v]_{\mathcal{B}}.$$

Then, you can check that B is a bilinear form and, moreover, since we have

$$[u]_{\mathcal{B}}^t X [v]_{\mathcal{B}} = B(u, v),$$

for every $u, v \in V$, then we must have that $X = [B]_{\mathcal{B}}$, by the defining property of $[B]_{\mathcal{B}}$.

5. Prove that every bilinear form $B \in \mathrm{Bil}_{\mathbb{K}}(\mathbb{K}^n)$ is of the form $B = B_A$, for some $A \in Mat_n(\mathbb{K})$.

Solution: Let $B \in \mathrm{Bil}_{\mathbb{K}}(\mathbb{K}^n)$. Consider the matrix

$$[B]_{\mathcal{S}^{(n)}} = A = [a_{ij}] \in Mat_n(\mathbb{K}).$$

Then, claim that $B = B_A$: indeed, let $u, v \in \mathbb{K}^n$. Then, we have that

$$B(u, v) = [u]_{\mathcal{S}^{(n)}}^t [B]_{\mathcal{S}^{(n)}} [v]_{\mathcal{S}^{(n)}} = u^t A v = B_A(u, v).$$

Hence, as the functions B and B_A agree for all inputs $(u, v) \in \mathbb{K}^n \times \mathbb{K}^n$, we must have that $B = B_A$.

6. Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$. Prove that if $\sigma_B : V \to V^*$ is injective then B is nondegenerate.

Solution: Suppose that σ_B is injective. Then, we have

$$\ker \sigma_B = \{v \in V \mid \sigma_B(v) \in V^*\} = \{0_V\}.$$

So, if $v \in V$ is such that $\sigma_B(v) = 0_{V^*}$ is the zero linear form (ie, for every $u \in V$ we have $(\sigma_B(v))(u) = 0 \in \mathbb{K}$) then $v = 0_V$.

Now, suppose that $v \in V$ is such that

$$B(u, v) = 0, \quad \text{for every } u \in V.$$

We want to show that $v = 0_V$: since

$$0 = B(u, v) \stackrel{def}{=} (\sigma_B(v))(u), \quad \text{for every } u \in V,$$

then we must have that $v = 0_V$, by injectivity of σ_B. Hence, we see that B is nondegenerate.

7. Prove the polarisation identity: if $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ is symmetric then, for every $u, v \in V$,

$$B(u, v) = \frac{1}{2}(B(u + v, u + v) - B(u, u) - B(v, v)).$$

8. Let $B \in \mathrm{Bil}_K(V)$ be antisymmetric. Prove that $B(u, u) = 0$, for every $u \in V$.

Solution: Let $u \in V$. Then, as B is antisymmetric we have that

$$B(u, u) = -B(u, u) \implies 2B(u, u) = 0 \in \mathbb{K} \implies B(u, u) = 0.$$

Since u is arbitrary this holds for every $u \in V$.