Math 110, Summer 2012 Short Homework 9

Due Thursday 7/26, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

0. Was this homework assignment too easy/too difficult/about right? Any other comments are welcome.

Calculations

1. Give an example of a nondegenerate antisymmetric bilinear form on $\mathbb{Q}^4$.

2. Determine the matrix of B with respect to the given basis $\mathcal{B}$. State whether the bilinear form is symmetric/antisymmetric/neither and if it is nondegenerate:

i) $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}\ ;\ (\underline{x}, \underline{y}) \mapsto x_1y_1 + 3x_2y_2 + y_3x_2 - 10x_3y_2,\ \mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right).$

ii) $B : Mat_3(\mathbb{Q}) \times Mat_3(\mathbb{Q}) \to \mathbb{Q}\ ;\ (X, Y) \mapsto \mathrm{tr}(XY),$

$$\mathcal{B} = (e_{11}, e_{12} - e_{21}, e_{32}, e_{13} - e_{31}, e_{13} + e_{31}, e_{22} + e_{33}, e_{33}, e_{23} - 2e_{32}, e_{12} + e_{11}).$$

3. Determine the adjoint of f with respect to B:

i) $B : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}\ ;\ (\underline{x}, \underline{y}) \mapsto \underline{x} \cdot \underline{y}, \quad f : \mathbb{R}^4 \to \mathbb{R}^4\ ;\ \underline{x} \mapsto A\underline{x}$, where

$$A = \begin{bmatrix} \pi & -1 & 0 & 0 \\ e^2 & \sqrt{2} & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -\sqrt{5} & 0 & 10 & 1 \end{bmatrix}.$$

ii) $B : Mat_2(\mathbb{Q}) \times Mat_2(\mathbb{Q}) \to \mathbb{Q}\ ;\ (X, Y) \mapsto \mathrm{tr}(XY), \quad f : Mat_2(\mathbb{Q}) \to Mat_2(\mathbb{Q})\ ;\ X \mapsto X^t.$

Proofs

4. Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ and $\mathcal{B} \subset V$ be an ordered basis. Prove that $[-]_{\mathcal{B}} : \mathrm{Bil}_{\mathbb{K}}(V) \to Mat_n(\mathbb{K})$ ($n = \dim V$) is linear and bijective.

5. Prove that every bilinear form $B \in \mathrm{Bil}_{\mathbb{K}}(\mathbb{K}^n)$ is of the form $B = B_A$, for some $A \in Mat_n(\mathbb{K})$.

6. Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$. Prove that if $\sigma_B : V \to V^*$ is injective then B is nondegenerate.

7. Prove the polarisation identity: if $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ is symmetric then, for every $u, v \in V$,

$$B(u, v) = \frac{1}{2}(B(u + v, u + v) - B(u, u) - B(v, v)).$$

8. Let $B \in \mathrm{Bil}_K(V)$ be antisymmetric. Prove that $B(u, u) = 0$, for every $u \in V$.