Math 110, Summer 2012 Short Homework 6

Due Monday 7/9, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Calculations

1. Consider the matrix

$$A = \begin{bmatrix} 3 & 2 & 2 \\ -2 & -1 & -2 \\ 1 & 1 & 2 \end{bmatrix}.$$

Determine $\chi_A(\lambda)$ and give the eigenvalues of A - there are exactly two distinct eigenvalues, λ_1, λ_2. What is the algebraic multiplicity of each eigenvalue?

Determine a basis of $E_{\lambda_1}, E_{\lambda_2}$, the eigenspaces of A. What is the geometric multiplicity of each eigenvalue? Explain why A is diagonalisable. Give an invertible matrix P such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{bmatrix}.$$

2. Consider the matrix

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \in Mat_3(\mathbb{C}).$$

Show that the subspace

$$U = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0 \right\} \subset \mathbb{C}^3,$$

is B-invariant and that 1 is an eigenvalue of B. Show that $E_1 \cap U = \{0_{\mathbb{C}^3}\}$. Find a B-invariant subspace $W \subset V$ such that

$$V = W \oplus U.$$

Justify your answer.

Proofs

3. Let V be a finite dimensional $\mathbb{C}$-vector space, $f \in \text{End}_{\mathbb{C}}$. Prove that 0 is an eigenvalue of f if and only if f is not injective.

4. Let $A \in Mat_5(\mathbb{C})$. Suppose that $\text{rank}A = 3$ and that A has three distinct nonzero eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{K}$. Prove that A is diagonalisable.

(*You need to use the information given to try and determine what $\chi_A(\lambda)$ looks like so that you can try and use Proposition 2.1.14.*)

5. Let $f \in \text{End}_{\mathbb{C}}(V)$ and $U \subset V$ an f-invariant subspace. of V. Prove:

- U is also $f^k = f \circ \cdots \circ f$-invariant.
- If U is also g-invariant, for some $g \in \text{End}_{\mathbb{C}}(V)$, then U is $(f+g)$-invariant.
- If $\lambda \in \mathbb{C}$ then U is λf-invariant.
- Prove that $\text{im} f, \ker f$ are f-invariant.

6. Let $f \in \text{End}_{\mathbb{C}}(V)$, with V an n-dimensional $\mathbb{C}$-vector space. Suppose that $f^2 = f \circ f = f$.

- Prove that $V = \text{im} f \oplus \ker f$.
- Prove that the only eigenvalues of f are $\lambda = 0, 1$.

  (*If λ is any eigenvalue, determine a polynomial relation on λ that forces $\lambda = 0, 1$.*)
- Deduce that $\chi_f(\lambda) = \lambda^s(1-\lambda)^{n-s}$, for some $1 \leq s < n$.
- Prove that $\text{im} f = E_1$ is the 1-eigenspace of f and deduce that $f = p_U$, for $U = \text{im} f$.

  (*Here p_U is the 'projection onto U morphism' discussed on p. 60 of the notes.*)