Math 110, Summer 2012 Short Homework 5 (SOME) SOLUTIONS

Due Thursday 7/5, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Calculations

1. Consider the bases

$$\mathcal{S}^{(2)} = (e_1, e_2) \subset \mathbb{Q}^2,\ \mathcal{C} = (f_1, f_2, f_3) \subset \mathbb{Q}^{\{1,2,3\}},$$

where

$$f_1(1) = 1, f_1(2) = 0, f_1(3) = -1;\ f_2(1) = 0, f_2(2) = -1, f_2(3) = -1;\ f_3(1) = 1, f_3(2) = 2, f_3(3) = 0.$$

Consider the following linear morphism

$$\alpha : \mathbb{Q}^2 \to \mathbb{Q}^{\{1,2,3\}}\ ;\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \alpha\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = f : \begin{cases} 1 \mapsto x_1 \\ 2 \mapsto x_2 \\ 3 \mapsto x_1 + x_2 \end{cases}.$$

Determine $[\alpha]^{\mathcal{C}}_{\mathcal{S}^{(2)}}$. Is α injective? Explain your answer. What is $[\alpha]^{\mathcal{B}}_{\mathcal{S}^{(2)}}$, where $B = \{e_i \mid i = 1, 2, 3\} \subset \mathbb{Q}^{\{1,2,3\}}$?

(*To determine $[\alpha]^{\mathcal{C}}_{\mathcal{S}^{(2)}}$ you will need to find the $\mathcal{C}$-coordinates of $\alpha(e_i)$ - to do this it may help to use the change of coordinate matrix $P_{\mathcal{C}\leftarrow\mathcal{B}} = P^{-1}_{\mathcal{B}\leftarrow\mathcal{C}}$.*)

Solution: Using the properties of associating matrices to morphisms we have

$$[\alpha]^{\mathcal{C}}_{\mathcal{S}^{(2)}} = [\mathrm{id}_{\mathbb{Q}^{\{1,2,3\}}} \circ \alpha]^{\mathcal{C}}_{\mathcal{S}^{(2)}} = [\mathrm{id}^{\{1,2,3\}}_{\mathbb{Q}}]^{\mathcal{C}}_{\mathcal{B}}[\alpha]^{\mathcal{B}}_{\mathcal{S}^{(2)}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\alpha]^{\mathcal{B}}_{\mathcal{S}^{(2)}}.$$

Then, we have

$$[\alpha]^{\mathcal{B}}_{\mathcal{S}^{(2)}} = [[\alpha(e_1)]_{\mathcal{B}}[\alpha(e_2)]_{\mathcal{B}}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix},$$

and

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ -1 & -1 & 0 \end{bmatrix}.$$

Then, we find that

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = P^{-1}_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & -2 \\ -1 & 1 & -1 \end{bmatrix}.$$

Hence, we have

$$[\alpha]^{\mathcal{C}}_{\mathcal{S}^{(2)}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\alpha]^{\mathcal{B}}_{\mathcal{S}^{(2)}} = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & -2 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -4 & -1 \\ -2 & 0 \end{bmatrix}.$$

We see that α is injective since there is a pivot in every column of $[\alpha]^{\mathcal{B}}_{\mathcal{S}^{(2)}}$ (or $[\alpha]^{\mathcal{C}}_{\mathcal{B}}$, either will do).

2. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \in Mat_{2,3}(\mathbb{Q}).$$

Determine the rank of A, rank $A = r$, and find matrices $P \in \mathrm{GL}_3(\mathbb{Q})$, $Q \in \mathrm{GL}_2(\mathbb{Q})$ such that

$$Q^{-1}AP = \begin{bmatrix} I_r & 0_{r,3-r} \\ 0_{2-r,r} & 0_{2-r,3-r} \end{bmatrix}.$$

(*Of course, if $r = 2$ then we do not have the bottom row. You need to replicate your proof of Q4.*)

Solution: We have rank $A = \dim \operatorname{im} T_A$ and since

$$Ae_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, Ae_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \in \operatorname{im} T_A,$$

and these vectors are linearly independent, we must have $\operatorname{im} T_A = \mathbb{Q}^2$ (we have just shown that $\operatorname{im} T_A$ is at least 2 dimensional, since it's a subspace of a 2 dimensional space it must be the whole space). Hence, rank $A = 2$.

There are many ways to proceed to find P, Q - you can use elementary matrices or the following procedure: we find a basis for $\ker \mathbf{T}_A \subset \mathbb{Q}^3$. By row-reducing A we see that any solution $\underline{x}$ such that $A\underline{x} = \underline{0}$ must take the form

$$\underline{x} = \begin{bmatrix} -3c \\ -c \\ c \end{bmatrix}, \quad \text{for some } c \in \mathbb{Q}.$$

Now, extend the basis

$$\mathcal{B} = (\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix},$$

of $\ker T_A$ to a basis $\mathcal{B}$ of $\mathbb{Q}^3$. For example, we can extend to

$$\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \right).$$

Then,

$$\mathcal{C}' = \left(A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right),$$

is a basis of $\operatorname{im} T_A$. Since $\operatorname{im} T_A = \mathbb{Q}^2$ we don't need to extend to a basis of $\mathbb{Q}^2$ so that we can take

$$Q = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

and we have

$$Q^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

3. Let $\mathcal{S}^{(3)} = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{Q}^3$. There are six possible orderings of $\mathcal{S}^{(3)}$: write them all down to obtain six different ordered bases $\mathcal{B}_1, \dots, \mathcal{B}_6$ and so that $\mathcal{B}_1 = (e_1, e_2, e_3)$. Write down the change of coordinates matrices $P_{\mathcal{B}_i \leftarrow \mathcal{B}_1}$, for $i = 1, \dots, 6$.

Proofs

4. Let $f, g \in \operatorname{Hom}_{\mathbb{K}}(V, W)$, $\mathcal{B} = \{b_1, \dots, b_n\} \subset V$ a basis of V. Prove that if $f(b_i) = g(b_i)$, for each $i = 1, \dots, n$, then $f = g$.

(*In order to show that two functions $f, g : V \to W$ are equal, you must show that $f(v) = g(v)$, for every $v \in V$. Therefore, this questions tells us that in order to show two linear morphisms are equal, it suffices to check that they are equal on a basis.*)

Solution: Let $v \in V$. Then, since $\mathcal{B}$ is a basis of V we have

$$v = \lambda_1 b_1 + \dots + \lambda_n b_n,$$

so that

$$\begin{aligned} f(v) &= f(\lambda_1 b_1 + \dots + \lambda_n b_n) \\ &= \lambda_1 f(b_1) + \dots + \lambda_n f(b_n) \\ &= \lambda_1 g(b_1) + \dots + \lambda_n g(b_n) \\ &= g(\lambda_1 b_1 + \dots + \lambda_n b_n) = g(v). \end{aligned}$$

Hence, we see that $f = g$.

5. Let $A \in Mat_{m,n}(\mathbb{K})$ be such that rank $A = r$. Prove that there exists $P \in \mathrm{GL}_n(\mathbb{K}), Q \in \mathrm{GL}_m(\mathbb{K})$ such that

$$Q^{-1}AP = \begin{bmatrix} I_r & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}.$$

(*Replicate the proof of Theorem 1.7.14 for $f = T_A$. How do P, Q arise?*)

Solution: Take a basis $\mathcal{B}' = (b_{r+1}, \dots, b_n)$ of ker T_A and extend to a basis $\mathcal{B}$ of $\mathbb{K}^n$, say

$$\mathcal{B} = (b_1, \dots, b_r, b_{r+1}, \dots, b_n).$$

Then, as in the proof of the Rank Theorem we see that

$$\mathcal{C}' = (Ab_1, \dots, Ab_r),$$

is a basis of $\mathrm{im}\, T_A$. Now, extend this to a basis

$$\mathcal{C} = (Ab_1, \dots, Ab_r, c_{r+1}, \dots, c_m),$$

of $\mathbb{K}^m$. Now, let

$$P = [b_1 \ \cdots \ b_n], \ Q = [Ab_1 \ \cdots \ c_m].$$

The result follows by relating that matrix of T_A with respect to $\mathcal{B}$ and $\mathcal{C}$ and the matrix of T_A with respect to the standard bases.