Math 110, Summer 2012 Short Homework 4 (SOME) SOLUTIONS

Due Monday 7/2, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Calculations

1. Which of the following subsets are bases of the vector space V? Explain your answer.

$$A = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^3,$$

$$B = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ -2 & 2 \end{bmatrix} \right\} \subset Mat_2(\mathbb{Q}),$$

$$C = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\} \subset U = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0 \right\}.$$

Solution:

- Yes: A is a linearly independent subset of a 3 dimensional vector space with 3 vectors in it.
- No: since $Mat_2(\mathbb{Q})$ is 4 dimensional and B contains only three vectors it is not possible for B to be a basis (Basis Theorem).
- No: U is a 2 dimensional vector space ($U = \ker f$, where : $\mathbb{C}^3 \to \mathbb{C}$ is the 'sum all entries' morphism; now use Rank Theorem) and C has three vectors in it, so can't be a basis.

2. Consider the linear morphism

$$\text{tr} : Mat_2(\mathbb{R}) \to \mathbb{R} \; ; \; A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mapsto a_{11} + a_{12}.$$

Determine an ordered basis $\mathcal{B}$ of the subspace $U = \ker \text{tr} \subset Mat_2(\mathbb{R})$, and an ordered basis $\mathcal{C} \subset \mathbb{R}$ of $\mathbb{R}$ making sure to explain why you know that the ordered sets you give are bases.

Using the ordered bases $\mathcal{B}$ and $\mathcal{C}$ you have found, determine the matrix $[\text{tr}]_{\mathcal{B}}^{\mathcal{C}}$ of tr relative to $\mathcal{B}$ and $\mathbb{C}$.

Solution: An ordered basis of U is

$$\mathcal{B} = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right),$$

and an ordered basis of $\mathbb{R}$ is $\mathcal{C} = (1)$. Then, since for any $v \in \ker \text{tr}$ we have $\text{tr}(v) = 0$, we must have that $\text{tr}(b) = 0$, for each $b \in \mathcal{B}$. Hence, the matrix of tr with respect to $\mathcal{B}$ and $\mathcal{C}$ is

$$[\text{tr}]_{\mathcal{B}}^{\mathcal{C}} = [0\ 0\ 0].$$

3. Consider the vector subspace (you DO NOT have to show this)

$$S_n = \{A \in Mat_n(\mathbb{Q}) \mid A = A^t\} \subset Mat_n(\mathbb{Q}),$$

where A^t is the transpose of A (so that if $A = [a_{ij}]$ then $A^t = [a_{ji}]$). S_n consists of all *symmetric $n \times n$ matrices with $\mathbb{Q}$-entries.*

a) Determine a basis $\mathcal{B}$ of S_n and show that the subset you obtain is a basis.

b) Find a closed formula for the dimension of S_n.

(*It might help to consider what happens when $n = 2, 3, 4$ first*)

Consider the subspace

$$A_n = \{A \in Mat_n(\mathbb{Q}) \mid A = -A^t\} \subset Mat_n(\mathbb{Q}).$$

A_n consists of all *antisymmetric $n \times n$ matrices eith $\mathbb{Q}$-entries.*

c) Determine a basis $\mathcal{C}$ of A_n and show that the subset you obtain is a basis.

d) Find a closed formula for the dimension of A_n.

e) Show that $S_n \cap A_n = \{0_n\}$ and deduce that $Mat_n(\mathbb{Q}) = A_n \oplus S_n$.

f) You have just shown that $\mathcal{D} = \mathcal{B} \cup \mathcal{C}$ is a basis of $Mat_n(\mathbb{Q})$. Find the $\mathcal{D}$-coordinates of the matrix

$$P = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 2 \\ -1 & -2 & 1 \end{bmatrix} \in Mat_3(\mathbb{Q}).$$

Solution:

a) If $A = [a_{ij}] \in S_n$ then we must have $a_{ij} = a_{ji}$, for every i, j. Moreover, any $A = [a_{ij}] \in Mat_n(\mathbb{Q})$ such that $a_{ij} = a_{ji}$ is a symmetric matrix. Then, the following is an ordered basis

$$\mathcal{B} = (e_{11}, e_{22}, \dots, e_{nn}, e_{ji}; e_{ij} + e_{ji} \mid i < j) :$$

it is easy to see that this set is linearly independent and if $A = [a_{ij}] \in S_n$, so that $a_{ij} = a_{ji}$, then

$$A = a_{11}e_{11} + \dots + a_{nn}e_{nn} + \sum_{i<j} a_{ij}(e_{ij} + e_{ji}),$$

so that $\text{span}_{\mathbb{Q}}\mathcal{B} = S_n$. Hence, $\mathcal{B}$ is a basis.

b) We need to count the vectors in $\mathcal{B}$: we have n vectors coming from $e_{11}, \dots, e_{nn}$ and are left with counting the size of the set $\{e_{ij} + e_{ji} \mid 1 \leq i < j \leq n\}$. For $i = 1$ there are $n-1$ possibilities of j so that $i < j \leq n$, for $i = 2$ there are $n-2$ possibilities for $i < j \leq n$, for $i = 3$ there are $n-3$ possibilities for $i < j \leq n$ etc. So, for $i = k$ there are $n-k$ possibilities for $i < j \leq n$. Hence, we have an extra $(n-1) + (n-2) + \dots + 2 + 1$ vectors. Thus,

$$\dim_{\mathbb{Q}} S_n = 1 + 2 + 3 + \dots + (n-2) + (n-1) + n = \frac{1}{2}n(n+1).$$

c) If $A = [a_{ij}] \in A_n$ then $a_{ij} = -a_{ji}$; in particular, we must have $a_{ii} = 0$, for $i = 1, \dots, n$. Now, a basis is

$$\mathcal{C} = (e_{ij} - e_{ji} \mid 1 \leq i < j \leq n).$$

as you can check that $\mathcal{C}$ is linearly independent and spans $Mat_2(\mathbb{Q})$.

d) Counting as before we see that

$$\dim A_n = 1 + 2 + 3 + \dots + (n-2) + (n-1) = \frac{1}{2}n(n-1).$$

(*There are ne diagonal matrices this time.*)

e) Let $A = [a_{ij}] \in S_n \cap A_n$. Then, we must have that $a_{ij} = a_{ji}$, for every i, j, and $a_{ij} = -a_{ji}$, for every i, j. Hence, we must have

$$a_{ij} = a_{ji} = -a_{ij}, \quad \text{for every } i, j,$$

so that $a_{ij} = 0$, for every i, j, and $A = 0_n$. So, we know that sum $A_n + S_n = A_n \oplus S_n$ is direct. Now, use the dimension formula

$$\dim(A_n + S_n) = \dim A_n + \dim S_n - \dim A_n \cap S_n = n^2.$$

So, we must have $Mat_n(\mathbb{Q}) = A_n + S_n = A_n \oplus S_n$, since $\dim Mat_n(\mathbb{Q}) = n^2$.

f) Denote

$$\mathcal{D} = (e_{11}, e_{22}, e_{33}, e_{12} + e_{21}, e_{13} + e_{31}, e_{23} + e_{32}, e_{12} - e_{21}, e_{13} - e_{31}, e_{23} - e_{32}).$$

Then, if $\mathcal{S} = (e_{11}, e_{12}, e_{13}, e_{21}, e_{22}, e_{23}, e_{31}, e_{32}, e_{33})$ is the standard basis then we have

$$P_{\mathcal{S}\leftarrow\mathcal{D}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\implies P_{\mathcal{D}\leftarrow\mathcal{S}} = P_{\mathcal{S}\leftarrow\mathcal{D}}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & -1/2 & 0 \end{bmatrix}.$$

Hence,

$$[P]_{\mathcal{D}} = P_{\mathcal{D}\leftarrow\mathcal{S}}[P]_{\mathcal{S}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 2 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1/2 \\ -1 \\ 0 \\ -1/2 \\ 0 \\ 2 \end{bmatrix}.$$

Proofs

4. Let V be a $\mathbb{K}$-vector space, $\mathcal{B} = (b_1, \dots, b_n)$. Prove that

$$V = \bigoplus_{i=1}^{n} \operatorname{span}_{\mathbb{K}}\{b_i\} = \operatorname{span}_{\mathbb{K}}\{b_1\} \oplus \cdots \oplus \operatorname{span}_{\mathbb{K}}\{b_n\}.$$

(*You must show that* $V = \operatorname{span}_{\mathbb{K}}\{b_1\} + \dots + \operatorname{span}_{\mathbb{K}}\{b_n\}$ *and that this sum is direct*)

Solution: As $\mathcal{B}$ is a basis of V then we see that

$$V = \mathrm{span}_{\mathbb{K}}\{b_1\} + ... + \mathrm{span}_{\mathbb{K}}\{b_n\},$$

as every $v \in V$ can be written as a linear combination of vectors in $\mathcal{B}$. Moreover, this sum is direct: suppose that $x \in \mathrm{span}_{\mathbb{K}}\{b_i\} \cap (\sum_{j \neq i} \mathrm{span}_{\mathbb{K}}\{b_j\}$. Then,

$$x = a_i b_i, \quad \text{and} \quad x = \sum_{j \neq i} a_j b_j.$$

Hence, we have

$$a_i b_i = \sum_{j \neq i} a_j b_j \implies a_i b_i - \sum_{j \neq i} a_j b_j = 0_V \implies a_1 = a_2 = ... = a_n = 0.$$

The result follows.