Math 110, Summer 2012 Short Homework 3 (SOME) SOLUTIONS

Due Monday 6/27, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Calculations

1. Is the function

$$\alpha : \mathbb{Q}[t] \to \mathbb{Q}[t] \ ; \ f \mapsto t^3 f - 3t,$$

a $\mathbb{Q}$-linear morphism? Justify your answer. Here $\mathbb{Q}[t]$ is the $\mathbb{Q}$-vector space of polynomials defined in the notes.

Solution: No, α is not linear: we see that

$$\alpha(0_{\mathbb{Q}[t]}) = t^3.0_{\mathbb{Q}[t]} - 3t = -3t \neq 0_{\mathbb{Q}[t]}.$$

Since linear morphisms must take zero vectors to zero vectors we can't possibly have that α is linear.

2. Which of the following functions are $\mathbb{K}$-linear? Justify your answers.

$$f : \mathbb{R}^2 \mapsto \mathbb{R}^4 \ ; \ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 3x_1 + 2x_2 \\ \exp(x_1) \\ 0 \\ -x_1 \end{bmatrix}, \ (\mathbb{K} = \mathbb{R})$$

$$g : Mat_{3,2}(\mathbb{C}) \mapsto Mat_{2,3}(\mathbb{C}) \ ; \ A \mapsto PAQ, \text{ where } P, Q \in Mat_{2,3}(\mathbb{Q}) \text{ are fixed, } (\mathbb{K} = \mathbb{C})$$

$$h : \mathbb{Q}^{\{1,2,3\}} \mapsto Mat_{2,2}(\mathbb{Q}) \ ; \ (f : i \mapsto f(i)) \mapsto \begin{bmatrix} f(1) & 2f(2) + 3f(3) \\ 0 & -f(1) \end{bmatrix}. \ (\mathbb{K} = \mathbb{Q})$$

Solution:

f: No: we do not have $f(\underline{0}) = \underline{0}$.

g: Yes: this follows from basic properties of matrix arithmetic. For $A, B \in Mat_2(\mathbb{C}), \lambda \in \mathbb{C}$ we have

$$g(A+\lambda B) = P(A+\lambda B)Q = P(AQ+\lambda BQ) = PAQ+P(\lambda B)Q = PAQ+\lambda PBQ = g(A)+\lambda g(B).$$

Hence, g satisfies LIN so is linear.

h: Yes: let $f, g \in \mathbb{Q}^{\{1,2,3\}}, \lambda \in \mathbb{Q}$. Then,

$$\begin{aligned} h(f + \lambda g) &= \begin{bmatrix} (f + \lambda g)(1) & 2(f + \lambda g)(2) + 3(f + \lambda g)(3) \\ 0 & -(f + \lambda g)(1) \end{bmatrix} \\ &= \begin{bmatrix} f(1) + \lambda g(1) & 2f(2) + \lambda 2g(2) + 3f(3) + \lambda 3g(3) \\ 0 & -f(1) - \lambda g(1) \end{bmatrix} \\ &= \begin{bmatrix} f(1) & 2f(2) + 3f(3) \\ 0 & -f(1) \end{bmatrix} + \lambda \begin{bmatrix} g(1) & 2g(2) + 3g(3) \\ 0 & -g(1) \end{bmatrix} = h(f) + \lambda h(g). \end{aligned}$$

Hence, h satisfies LIN so is linear.

Proofs

3. Let P be the set of positive numbers, so $P = (0, \infty)$. Define

$$\alpha : P \times P \to P \ ; \ (x, y) \mapsto xy, \quad \sigma : \mathbb{R} \times P \to P \ ; \ (\lambda, x) \mapsto x^\lambda.$$

Show that (P, α, σ) is an $\mathbb{R}$-vector space. You must check Axioms VS1-VS8 and you need to define $0_P \in P$ and, for any $x \in P$, $-x \in P$.

Can you explain how this 'weird' $\mathbb{R}$-vector space arises? (*Hint: there is a bijective function $L : P \to \mathbb{R}$ that might help you understand why we have defined 'addition' as 'multiplication'.*)

Solution: We must have $0_P = 1 \in P$ and, for any $x \in P$, we take $-x = x^{-1} \in P$. Then, all of the axioms hold.

The function $\log : P \to \mathbb{R}$ is such that $\log(xy) = \log(x) + \log(y)$ and so turns multiplication into addition. This is where this $\mathbb{R}$-vector space structure comes from.

4. Let V be a $\mathbb{K}$-vector space, $E \subset V$ a nonempty subset. Prove that $\text{span}_{\mathbb{K}} E$ is equal to the intersection of all subspaces $U \subset V$ such that $E \subset U$. So, if $\mathcal{F}$ is the set of all subspaces of V that contain E (ie, $U \in \mathcal{F}$ if and only if $E \subset U$), then prove that

$$\text{span}_{\mathbb{K}} E = \bigcap_{U \in \mathcal{F}} U.$$

(*Hint: to show that two sets A, B are equal, it suffices to show that $A \subset B$ and $B \subset A$.*)

Solution: $\text{span}_{\mathbb{K}} E \subset \bigcap_{U \in \mathcal{F}} U$: let $v = c_1 e_1 + ... + c_n e_n \in \text{span}_{\mathbb{K}} E$. Then, since $E \subset U$, for every $U \in \mathcal{F}$, and each U is a subspace (so closed under addition and scalar multiplication), we must have $v \in U$, for every $U \in \mathcal{F}$. Hence, $\text{span}_{\mathbb{K}} E \subset \bigcap_{U \in \mathcal{F}} U$.

$\bigcap_{U \in \mathcal{F}} U \subset \text{span}_{\mathbb{K}} E$: since $\text{span}_{\mathbb{K}} E$ is a subspace of V and $E \subset \text{span}_{\mathbb{K}} E$, we have that $\text{span}_{\mathbb{K}} E \in \mathcal{F}$. Now, for every $W \in \mathcal{F}$ we have that

$$\bigcap_{U \in \mathcal{F}} U \subset W,$$

because, by definition,

$$\bigcap_{U \in \mathcal{F}} U = \{v \in V \mid v \in U, \text{ for every } U \in \mathcal{F}\},$$

so that vectors in $\bigcap_{U \in \mathcal{F}} U$ are, by definition, vectors that lie in every $U \in \mathcal{F}$. In particular, for a specific $W \in \mathcal{F}$, vectors in $\bigcap_{U \in \mathcal{F}} U$ must lie in W, so that $\bigcap_{U \in \mathcal{F}} U$ is a subset of W. Hence, for $W = \text{span}_{\mathbb{K}} E \in \mathcal{F}$, we have

$$\bigcap_{U \in \mathcal{F}} U \subset \text{span}_{\mathbb{K}} E.$$

5. Let V, W be $\mathbb{K}$-vector spaces, $f \in \text{Hom}_{\mathbb{K}}(V, W)$ an isomorphism. Let E be a nonempty subset of V. Prove:

- E is linearly independent in V if and only if $f(E)$ is linearly independent in W.
- E spans V if and only if $f(E)$ spans W.

Here, we define $f(E) = \{f(e) \mid e \in E\}$.

Solution:

- ($\Rightarrow$) Suppose that E is linearly independent. We want to show that $f(E)$ is linearly independent. So, suppose that there is a linear relation

  $$c_1 f(e_1) + ... + c_n f(e_n) = 0_W,$$

  among vectors in $f(E)$. Then, as f is linear we have

  $$f(c_1 e_1 + ... + c_n e_n) = 0_W,$$

  so that

  $$c_1 e_1 + ... + c_n e_n \in \ker f = \{0_V\}, \quad \text{since } f \text{ injective.}$$

  Hence, we have

  $$c_1 e_1 + ... + c_n e_n = 0_V,$$

  so that $c_1 = ... = c_n = 0$ since E is assumed linearly independent.

  ($\Leftarrow$) Suppose that $f(E)$ is linearly independent. Consider a linear relation among vectors in E and apply f to this linear relation to obtain a linear relation among vectors in $f(E)$. As $f(E)$ is linearly independent this must be the trivial linear relation so that the initial linear relation is also trivial. The details are left to the reader.

- ($\Rightarrow$) Suppose that E spans V. Then, since f is surjective we must have that $f(E)$ spans W. The details are left to the reader.

  ($\Leftarrow$) Suppose that $f(E)$ spans W, so that $\text{span}_{\mathbb{K}}\{f(e) \mid e \in E\} = W$. Let $v \in V$. Then, $f(v) \in W$ so that we have a linear combination

$$f(v) = c_1 f(e_1) + ... + c_n f(e_n) = f(c_1 e_1 + ... + c_n e_n).$$

  Since f is injective then we must have

$$v = c_1 e_1 + ... + c_n e_n \in \text{span}_{\mathbb{K}} E.$$

  Hence, $V = \text{span}_{\mathbb{K}} E$.