Math 110, Summer 2012 Short Homework 2, (SOME) SOLUTIONS

Due Monday 6/25, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Calculations

1. Determine the linear (in)dependence of the following subsets:

$$E_1 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \subset \mathbb{Q}^3,$$

$$E_2 = \left\{ I_2, A, A^2 \right\} \subset M_2(\mathbb{R}), \quad \text{where } A = \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{bmatrix}, \; I_2 \text{ is the } 2 \times 2 \text{ identity matrix.}$$

Solution: E_1 is linearly dependent since it is a subset containing 4 vectors in a 3-dimensional vector space. You can also show that E_1 is linearly dependent directly: form the matrix P whose columns are the columns vectors in E_1. Then,

$$P \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so that the solutions to the matrix equation $P\underline{x} = \underline{0}$ are those vectors in the set

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{Q}^4 \mid \begin{array}{rl} x_1 + 2x_3 & = 0 \\ x_2 + x_3 & = 0 \\ x_4 & = 0 \end{array} \right\} = \left\{ \begin{bmatrix} 2x \\ x \\ -x \\ 0 \end{bmatrix} \mid x \in \mathbb{Q} \right\}.$$

Hence, we have a linear relation

$$2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} = 0_{\mathbb{Q}^3}.$$

For the subset E_2 we first see that

$$A^2 = \begin{bmatrix} 1 & 2\sqrt{2} \\ 0 & 1 \end{bmatrix}.$$

Now to determine the linear (in)dependence of E_2 we need to consider a linear relation

$$0_{Mat_2(\mathbb{R})} = c_1 I_2 + c_2 A + c_3 A^2 = \begin{bmatrix} c_1 + c_2 + c_3 & \sqrt{2}c_2 + 2\sqrt{2}c_3 \\ 0 & c_1 + c_2 + c_3 \end{bmatrix}.$$

Thus, we have the system of linear equations

$$\begin{array}{rl} c_1 + c_2 + c_3 & = 0 \\ \sqrt{2}c_2 + 2\sqrt{2}c_3 & = 0' \end{array}$$

with coefficient matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2} & 2\sqrt{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

There is a solution

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix},$$

of this system, implying that we have a nontrivial linear relation

$$0_{Mat_2(\mathbb{R})} = -I_2 + 2A - A^2.$$

Hence, E_2 is linearly dependent.

2. Find a vector $v \in E$ such that $\text{span}_{\mathbb{K}} E = \text{span}_{\mathbb{K}} E'$, where

$$E = \{I_2, B, B^2, B^3\} \subset Mat_2(\mathbb{Q}), \quad \text{where } B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

and $E' = E \setminus \{v\}$.

Solution: We have

$$B^2 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \; B^3 = \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix}.$$

Then, we have the linear relation

$$B^2 - B^3 = 2I_2.$$

Hence, we can remove the vector B^2 from the set E and we will have $\text{span}_{\mathbb{Q}} E = \text{span}_{\mathbb{Q}} E \setminus \{B^2\}$. This follows from the proof of the Elimination Lemma. We could also remove B^3 or I_2.

3. Let $V = \mathbb{R}^3$. Consider two planes $\Pi_1, \Pi_2 \subset \mathbb{R}^3$ that pass through the origin. Consider the corresponding vector subspaces $U_1, U_2 \subset \mathbb{R}^3$. Under what conditions must we have $U_1 + U_2 = \mathbb{R}^3$? Is it possible for $U_1 \cap U_2 = \{0_{\mathbb{R}^3}\}$?

Suppose that $W = \text{span}_{\mathbb{K}}\{v\} \subset \mathbb{R}^3$, for $v \in \mathbb{R}^3$. Under what conditions can we have $U_1 + W = \mathbb{R}^3$? Is it possible for $\mathbb{R}^3 = U_1 \oplus W$? Explain your answer.

Proofs

4. Let V be a $\mathbb{K}$-vector space, $U, W \subset V$ vector subspaces of V. Prove:

- $U + W$ is a vector subspace of V,
- $U \cap W$ is a vector subspace of V,
- $U \cup W$ is a vector subspace if and only if $U \subset W$ or $W \subset U$.

Give an example of two subspaces of $U, W \subset \mathbb{R}^2$ such that $U \cup W$ is not a subspace of $\mathbb{R}^2$.

Solution: We will show that each of the subspaces satisfies the SUB axiom:

- let $z_1, z_2 \in U + W$, $\lambda, \mu \in \mathbb{K}$, then there are $u_1, u_2 \in U$, $w_1, w_2 \in W$ such that $z_1 = u_1 + w_1, z_2 = u_2 + w_2$. Hence,

$$\begin{aligned} \lambda z_1 + \mu z_2 &= \lambda(u_1 + w_1) + \mu(u_2 + w_2) \\ &= (\lambda u_1 + \mu u_2) + (\lambda w_1 + \mu w_2) \in U + W, \end{aligned}$$

  since U and W are subspaces. Hence, $U + W$ is also a subspace.

- Let $v, u \in U \cap W$, $\lambda, \mu \in \mathbb{K}$. So, $u, v \in U$ and $u, v \in W$. Then, $\lambda u + \mu v \in U$ since U is a subspace, and $\lambda u + \mu v \in W$ since W is a subspace. Hence, by definition of $U \cap W$, we have that $\lambda u + \mu v \in U \cap W$ so that $U \cap W$ is a subspace.

- Suppose that $U \cup W$ is a subspace and that U is not a subset of W. We are going to show that $W \subset U$. Let $w \in W$ and $u \in U$ be such that $u \notin W$ (such u exists because we are assuming that U is not a subset of W). Then, $w + u \in U \cup W$ since $U \cup W$ is a subspace. Therefore, either $u + w \in U$ or $u + w \in W$ (or, perhaps, both U and W). We can't have $u + w \in W$, otherwise we will obtain $u \in W$, contradicting our choice of u. Hence, we must have $u + w \in U$ which implies that $w \in U$. Hence, $W \subset U$.

  Conversely, if $U \subset W$ or $W \subset U$ then $U \cup W$ equals either W or U, which is a subspace in either case.

For the example, take two distinct lines $L_1, L_2 \subset \mathbb{R}^2$. Then, $L_1 \cup L_2$ is the union of two lines and if we choose any nonzero $u_1 \in L_1$ and $u_2 \in L_2$, then $u_1 + u_2 \notin L_1 \cup L_2$, so that this set is not closed under addition.

5. Let V be a $\mathbb{K}$-vector space and $A, B \subset V$ be nonempty subsets of V. Prove:

$$\mathrm{span}_{\mathbb{K}}(A \cup B) = \mathrm{span}_{\mathbb{K}} A + \mathrm{span}_{\mathbb{K}} B.$$

6. Let $f \in \mathrm{Hom}_{\mathbb{K}}(V, W)$, where V, W are $\mathbb{K}$-vector spaces. Prove:

   - $\ker f \subset V$ is a vector subspace of V,
   - $\mathrm{im} f \subset W$ is a vector subspace of W.

Solution:

- let $u, v \in \ker f$, $\lambda, \mu \in \mathbb{K}$. Then, we have $f(u) = f(v) = 0_W$, by definition of $\ker f$. Hence, $f(\lambda u + \mu v) = \lambda f(u) + \mu f(v)$, since f is linear. Therefore, $f(\lambda u + \mu v) = 0_W + 0_W = 0_W$. Hence, $\lambda u + \mu v \in \ker f$, so that $\ker f$ is a subspace of V.
- let $w, z \in \mathrm{im} f$, $\lambda, \mu \in \mathbb{K}$. Then, there exists $u, v \in V$ such that $f(u) = w, f(v) = z$, by definition of $\mathrm{im} f$. Then, $f(\lambda u + \mu v) = \lambda f(u) + \mu f(v) = \lambda w + \mu z$. Hence, $\lambda w + \mu z \in \mathrm{im} f$ so that $\mathrm{im} f$ is a subspace of W.