Math 110, Summer 2012 Short Homework 10 (SOME) SOLUTIONS

Due Monday 7/30, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

0. Was this homework assignment too easy/too difficult/about right? Any other comments are welcome.

Calculations

1. For the following symmetric matrices $A \in \mathrm{GL}_n(\mathbb{R})$ determine $P \in \mathrm{GL}_n(\mathbb{R})$ such that

$$P^t A P = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \quad d_i \in \{1, -1\}.$$

i) $A = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$,

ii) $A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$,

iii) $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

What is the signature of each of the corresponding bilinear forms $B_A \in \mathrm{Bil}_{\mathbb{R}}(\mathbb{R}^n)$.

Solution: We use the 'completing the square method' from Example 3.2.9:

i) We have

$$\underline{x}^t A \underline{x} = x_1^2 - 2x_1x_2 = (x_1 - x_2)^2 + x_2^2.$$

Consider the new coordinates

$$y_1 = x_1 - x_2, \; y_2 = x_2,$$

so that

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (= Q\underline{x}).$$

If we set

$$P = Q^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

then we have

$$P^t A P = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}.$$

We see that

$$\mathrm{sig}(B_A) = 0.$$

ii) We have

$$\underline{x}^t A \underline{x} = -2x_1x_2 + 2x_1x_3 + x_3^2 = -(x_1 + x_2)^2 + x_2^2 + (x_1 + x_3)^2.$$

Consider the new coordinates

$$y_1 = x_1 + x_2, \; y_2 = x_2, \; y_3 = x_1 + x_3,$$

so that

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (= Q\underline{x}).$$

Set

$$P = Q^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix},$$

and we then have

$$P^t A P = \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix}.$$

We have

$$\operatorname{sig}(B_A) = 1.$$

iii) We have

$$\underline{x}^t A \underline{x} = x_1^2 - x_2^2 + 2x_1x_3 + 4x_2x_3 + 2x_3x_4 + 2x_4^2 = (x_1 + x_3)^2 - (x_2 - 2x_3)^2 + 3(x_3 + \frac{1}{3}x_4)^2 + \frac{5}{3}x_4^2.$$

Consider the new coordinates

$$y_1 = x_1 + x_3,\ y_2 = x_2 - 2x_3,\ y_3 = \sqrt{3}(x_3 + \frac{1}{3}x_4),\ y_4 = \sqrt{\frac{5}{3}}x_4,$$

so that

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & \sqrt{\frac{5}{3}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad (= Q\underline{x})$$

If we set

$$P = Q^{-1} = \begin{bmatrix} 1 & 0 & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{15}} \\ 0 & 1 & \frac{2}{\sqrt{3}} & \frac{-2}{\sqrt{15}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{15}} \\ 0 & 0 & 0 & \sqrt{\frac{3}{5}} \end{bmatrix},$$

then we have

$$P^t A P = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Hence, we have

$$\operatorname{sig}(B_A) = 2.$$

2. For the matrix A in *ii*) above, determine an ordered basis $\mathcal{B}$ of $\mathbb{C}^3$ such that

$$[B_A]_{\mathcal{B}} = I_3.$$

(*Hint: proceed as you would in the real case, except now you can use the fact that you are allowed to find square roots of negative numbers. See the proof of the classification of nondegenerate symmetric bilinear forms over $\mathbb{C}$ in Section 3.2.*)

Solution: Consider the coordinates

$$y_1 = \sqrt{-1}(x_1 + x_1),\ y_2 = x_2,\ y_3 = x_1 + x_3,$$

so that

$$\underline{x}^t A \underline{x} = y_1^2 + y_2^2 + y_3^2.$$

We can express these new coordinates as follows:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \sqrt{-1} & \sqrt{-1} & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \ (= Q\underline{x}).$$

Then, $Q = P_{\mathcal{B}\leftarrow\mathcal{S}^{(3)}}$, where $\mathcal{B}$ is such that

$$[B_A]_{\mathcal{B}} = P^t_{\mathcal{S}^{(3)}\leftarrow\mathcal{B}}[B_A]_{\mathcal{S}^{(3)}}P_{\mathcal{S}^{(3)}\leftarrow\mathcal{B}} = (Q^{-1})^t A Q^{-1}.$$

As

$$\underline{x}^t A \underline{x} = \underline{y}^t \underline{y} = (Q\underline{x})^t(Q\underline{x}) = \underline{x}^t Q^t Q \underline{x},$$

then we must have

$$A = Q^t Q,$$

so that

$$[B_A]_{\mathcal{B}} = I_3.$$

Hence, if we know $Q^{-1} = P_{\mathcal{S}^{(3)}\leftarrow\mathcal{B}}$ we can determine $\mathcal{B}$. So, we find that

$$Q^{-1} = \begin{bmatrix} -\sqrt{-1} & -1 & 0 \\ 0 & 1 & 0 \\ \sqrt{-1} & 1 & 1 \end{bmatrix}.$$

Hence,

$$\mathcal{B} = \left(\begin{bmatrix} -\sqrt{-1} \\ 0 \\ \sqrt{-1} \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

is such that

$$[B_A]_{\mathcal{B}} = I_3.$$

Proofs

3. Prove: let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$ be nondegenerate and symmetric, where $\mathbb{K}$ is ANY number field. Then, there exists an basis $\mathcal{B} \subset V$ such that

$$[B]_{\mathcal{B}} = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}, \ d_i \in \mathbb{K}.$$

(*This is a generalisation of the results of section 3.2. You just need to copy the proofs of the theorems in that section.*

Deduce that for any symmetric $A \in \mathrm{GL}_n(\mathbb{K})$ there is $P \in \mathrm{GL}_n(\mathbb{K})$ such that $P^t A P$ is diagonal.

Solution:

4. Let $B \in \mathrm{Bil}_{\mathbb{K}}(V)$. Prove that B can be written uniquely as $B = B_s + B_a$, where $B_s \in \mathrm{Bil}_{\mathbb{K}}(V)$ is symmetric and $B_a \in \mathrm{Bil}_{\mathbb{K}}(V)$ is antisymmetric.

(*Hint: you will need to use that $Mat_n(\mathbb{K}) = S_n \oplus A_n$ (recall SH4, Q3).*)

Solution: Recall the result from SH4: we have the direct sum decomposition

$$Mat_n(\mathbb{K}) = S_n \oplus A_n,$$

where

$$S_n = \{A \in Mat_n(\mathbb{K}) \mid A^t = A\}, \ A_n = \{A \in Mat_n(\mathbb{K}) \mid A = -A^t\}.$$

Let $\mathcal{B} \subset V$ be an ordered basis. Then, we have an isomorphism of vector spaces

$$[-]_{\mathcal{B}} : \mathrm{Bil}_{\mathbb{K}}(V) \to Mat_n(\mathbb{K}) = S_n \oplus A_n,$$

so for any $B \in \mathrm{Bil}_K(V)$ we have unique $X_s \in S_n, X_a \in A_n$ such that

$$[B]_{\mathcal{B}} = X_s + X_a.$$

If we define the bilinear forms

$$B_s(u, v) = [u]_{\mathcal{B}}^t X_s [v]_{\mathcal{B}},\ B_a(u, v) = [u]_{\mathcal{B}}^t X_a [v]_{\mathcal{B}},$$

then we have

$$B = B_s + B_a,$$

as

$$[B_s + B_a]_{\mathcal{B}} = [B_s]_{\mathcal{B}} + [B_a]_{\mathcal{B}} = X_s + X_a = [B]_{\mathcal{B}}.$$