Math 110, Summer 2012 Long Homework 4 (SOME) SOLUTIONS

Due Wednesday 8/1, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Please write your answers in complete English sentences (where applicable). Make your arguments rigorous - if something is 'obvious', state why this is the case. Full credit will be awarded to those solutions that are complete and answer the question posed in a coherent manner.

1. In this problem we will study (the mathematical version of) the Casimir operator of the angular momentum algebra utilised in quantum mechanics. Casimir operators are used to represent angular momentum, elementary particle mass and spin and many other important quantities that arise in the study of elementary particles.

Let

$$\mathfrak{so}(3) = \{X \in Mat_3(\mathbb{R}) \mid X = -X^t\} \subset Mat_3(\mathbb{R}).$$

This is the *Lie algebra of the special orthogonal group* $SO(3)$ - here

$$SO(3) = \{g \in O(3) \mid \det g = 1\}.$$

It is possible to identify $SO(3)$ as the set of all rigid rotations of $\mathbb{R}^3$ (ie, all possible rotations of the sphere). Essentially, elements of $\mathfrak{so}(3)$ are the *infinitesimal generators* of rigid rotations.

a) Show that $\mathfrak{so}(3)$ is a subspace of $Mat_3(\mathbb{R})$ and that $\mathcal{B} = (L_x, L_y, L_z)$ is a basis of $\mathfrak{so}(3)$, where

$$L_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \ L_y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ L_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Deduce that $\dim \mathfrak{so}(3) = 3$.

Solution: Let $X, Y \in \mathfrak{so}(3), \lambda, \mu \in \mathbb{R}$. Then,

$$(\lambda X + \mu Y)^t = \lambda X^t + \mu Y^t = -\lambda X^t - \mu Y^t = -(\lambda X + \mu Y)^t,$$

so that $\lambda X + \mu Y \in \mathfrak{so}(3)$ and $\mathfrak{so}(3)$ is a subspace.

Suppose that

$$a_x L_x + a_y L_y + a_z L_z = 0_3 \implies \begin{bmatrix} 0 & -a_z & -a_y \\ a_z & 0 & -a_x \\ a_y & a_x & 0 \end{bmatrix} = 0_3 \implies a_x = a_y = a_z = 0.$$

Hence, $\mathcal{B}$ is linearly independent.

Now, let $X = [x_{ij}] \in \mathfrak{so}(3)$. Then, $X = -X^t$, so that

$$X = \begin{bmatrix} 0 & x_{12} & x_{13} \\ -x_{12} & 0 & x_{23} \\ -x_{13} & -x_{23} & 0 \end{bmatrix} = -x_{12} L_z - x_{13} L_y - x_{23} L_x \in \text{span}_{\mathbb{R}} \mathcal{B}.$$

Hence, $\text{span}_{\mathbb{R}} \mathcal{B} = \mathfrak{so}(3)$ and $\mathcal{B}$ is a basis. Since $\mathcal{B}$ has 3 elements then $\dim \mathfrak{so}(3) = 3$.

b) Show that if $X, Y \in \mathfrak{so}(3)$ then

$$\text{ad}(X)(Y) \stackrel{def}{=} XY - YX \in \mathfrak{so}(3).$$

Hence, we have defined a function

$$\text{ad}(X) : \mathfrak{so}(3) \to \mathfrak{so}(3) \ ; \ Y \mapsto \text{ad}(X)(Y) = XY - YX.$$

Show that $\mathrm{ad}(X)$ is $\mathbb{R}$-linear, so that $\mathrm{ad}(X) \in \mathrm{End}_{\mathbb{R}}(\mathfrak{so}(3))$.

Solution: Let $X, Y \in \mathfrak{so}(3)$. Then,

$$(XY - YX)^t = (XY)^t - (YX)^t = Y^tX^t - X^tY^t = (-Y)(-X) - (-X)(-Y) = -(XY - YX).$$

Hence, $\mathrm{ad}(X)(Y) \in \mathfrak{so}(3)$. Let $Y, Z \in \mathfrak{so}(3), \lambda \in \mathbb{R}$. Then,

$$\mathrm{ad}(X)(Y+\lambda Z) = X(Y+\lambda Z)-(Y+\lambda Z)X = XY-YX+\lambda(XZ-ZX) = \mathrm{ad}(X)(Y)+\lambda\mathrm{ad}(X)(Z),$$

so that $\mathrm{ad}(X)$ is $\mathbb{R}$-linear.

b) Determine the 3×3 matrices

$$[\mathrm{ad}(L_x)]_{\mathcal{B}},\ [\mathrm{ad}(L_y)]_{\mathcal{B}},\ [\mathrm{ad}(L_z)]_{\mathcal{B}}.$$

Solution: We have

$$[\mathrm{ad}(L_x)]_{\mathcal{B}} = [[\mathrm{ad}(L_x)(L_x)]_{\mathcal{B}}[\mathrm{ad}(L_x)(L_y)]_{\mathcal{B}}[\mathrm{ad}(L_x)(L_z)]_{\mathcal{B}}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = -L_x,$$

$$[\mathrm{ad}(L_y)]_{\mathcal{B}} = [[\mathrm{ad}(L_y)(L_x)]_{\mathcal{B}}[\mathrm{ad}(L_y)(L_y)]_{\mathcal{B}}[\mathrm{ad}(L_y)(L_z)]_{\mathcal{B}}] = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = L_y,$$

$$[\mathrm{ad}(L_z)]_{\mathcal{B}} = [[\mathrm{ad}(L_z)(L_x)]_{\mathcal{B}}[\mathrm{ad}(L_z)(L_y)]_{\mathcal{B}}[\mathrm{ad}(L_z)(L_z)]_{\mathcal{B}}] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -L_z.$$

c) Consider the bilinear form (you do NOT have to show B is a bilinear form)

$$B : \mathfrak{so}(3) \times \mathfrak{so}(3) \to \mathbb{R}\ ;\ (X, Y) \mapsto B(X, Y) = \mathrm{tr}([\mathrm{ad}(X)]_{\mathcal{B}}[\mathrm{ad}(Y)]_{\mathcal{B}}).$$

i) Prove that, for every $U, V \in Mat_3(\mathbb{R})$ we have $\mathrm{tr}(UV) = \mathrm{tr}(VU)$. Deduce that B is symmetric.

ii) Prove that B is nondegenerate.

iii) For L_x, L_y, L_z find $M_x, M_y, M_z \in \mathfrak{so}(3)$ such that

$$B(L_i, M_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \qquad \text{Here we take } i, j \in \{x, y, z\}.$$

Solution:

i) Let

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix},\ V = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}.$$

By considering UV and VU you will find that

$$\mathrm{tr}(UV) = \mathrm{tr}(VU).$$

Hence, we have

$$B(X, Y) = \mathrm{tr}([\mathrm{ad}(X)]_{\mathcal{B}}[\mathrm{ad}(Y)]_{\mathcal{B}}) = \mathrm{tr}([\mathrm{ad}(Y)]_{\mathcal{B}}[\mathrm{ad}(X)]_{\mathcal{B}}) = B(Y, X),$$

so that B is symmetric.

ii) Consider the matrix

$$[B]_{\mathcal{B}} = [b_{ij}], \quad \text{where } b_{ij} = B(L_i, L_j), \text{ for } i, j \in \{x, y, z\}.$$

Then, we have

$$B(L_x, L_x) = \mathrm{tr}(L_x L_x) = -2, B(L_x, L_y) = \mathrm{tr}(-L_x L_y) = 0, B(L_x, L_z) = \mathrm{tr}(L_x L_z) = 0,$$

$$B(L_y, L_y) = \mathrm{tr}(L_y L_y) = -2, B(L_y, L_z) = \mathrm{tr}(-L_y L_z) = 0, B(L_z, L_z) = \mathrm{tr}(L_z L_z) = -2,$$

so that

$$[B]_{\mathcal{B}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Since this matrix is invertible then we have that B is nondegenerate.

iii) Determining M_x such that

$$B(L_i, M_x) = \begin{cases} 1, & i = x \\ 0, & i \neq x \end{cases}$$

is the same as determining $[M_x]_{\mathcal{B}} = \begin{bmatrix} a_{xx} \\ a_{xy} \\ a_{xz} \end{bmatrix}$ such that

$$[L_i]^t_{\mathcal{B}} [B]_{\mathcal{B}} [M_x]_{\mathcal{B}} = \begin{cases} 1, & i = x \\ 0, & i \neq x \end{cases} \quad .$$

Since $[L_x]_{\mathcal{B}} = e_1, [L_y]_{\mathcal{B}} = e_2, [L_z]_{\mathcal{B}} = e_3$, the above conditions imply that

$$[M_x]_{\mathcal{B}} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix}.$$

Hence, we have

$$M_x = -\frac{1}{2} L_x.$$

Similarly, we find that

$$M_y = -\frac{1}{2} L_y, \; M_z = -\frac{1}{2} L_z.$$

d) Consider the matrix

$$\Omega = L_x M_x + L_y M_y + L_z M_z \in Mat_3(\mathbb{R}).$$

Show that Ω is a scalar multiple of I_3.

Ω is called the *Casimir operator* and appears in the study of quantum angular momentum and leads to the definition of the quantum angular momentum number l (it is usually denoted L^2 or J^2 in QM textbooks). Our definition is slightly different than the one you'd see in QM, but only differs by a scalar multiple.

Solution: We have

$$\Omega = -\frac{1}{2}\left(L_x^2 + L_y^2 + L_z^2\right) = I_3.$$

2. Consider the basis $\mathcal{B} = (e, h, f) \subset sl_2(\mathbb{C})$, defined in Long Homework 1. Here

$$sl_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in Mat_2(\mathbb{C}) \right\}.$$

Following the steps in Questions 1 (in particular, b), c), d); you DO NOT have to do a) or c)i)) and using the basis $\mathcal{B}$ show that the Casimir operator of $sl_2(\mathbb{C})$ is

$$\Omega = h^2 + 2fe + 2ef \in Mat_2(\mathbb{C}).$$

Deduce, that Ω is a scalar multiple of I_2.

Solution: Set $\mathcal{B} = (e, h, f)$, where

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

b) We have

$$[\mathrm{ad}(e)]_{\mathcal{B}} = [[\mathrm{ad}(e)(e)]_{\mathcal{B}}[\mathrm{ad}(e)(h)]_{\mathcal{B}}[\mathrm{ad}(e)(f)]_{\mathcal{B}}] = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \stackrel{def}{=} E,$$

$$[\mathrm{ad}(f)]_{\mathcal{B}} = [[\mathrm{ad}(f)(e)]_{\mathcal{B}}[\mathrm{ad}(f)(h)]_{\mathcal{B}}[\mathrm{ad}(f)(f)]_{\mathcal{B}}] = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \stackrel{def}{=} F,$$

$$[\mathrm{ad}(h)]_{\mathcal{B}} = [[\mathrm{ad}(h)(e)]_{\mathcal{B}}[\mathrm{ad}(h)(h)]_{\mathcal{B}}[\mathrm{ad}(h)(f)]_{\mathcal{B}}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \stackrel{def}{=} H.$$

c)
  i) This is exactly the same proof as in the previous problem.
  ii) Consider the matrix

$$[B]_{\mathcal{B}} = [b_{ij}], \quad \text{where } b_{ij} = B(i, j), i, j \in \{e, h, f\}.$$

Then, we have

$$B(e, e) = \mathrm{tr}(EE) = 0,\ B(e, h) = \mathrm{tr}(EH) = 0,\ B(e, f) = \mathrm{tr}(EF) = 4,$$

$$B(h, h) = \mathrm{tr}(HH) = 8,\ B(h, f) = \mathrm{tr}(HF) = 0,\ B(f, f) = \mathrm{tr}(FF) = 0.$$

Hence, we have

$$[B]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{bmatrix}.$$

Therefore, as this matrix is invertible then we have that B is nondegenerate.

  iii) Proceeding as above, we want to find M_e, M_h, M_f such that

$$B(i, M_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases} \qquad \text{Here we have } i, j \in \{e, h, f\}.$$

This is the same as determining $[M_e]_{\mathcal{B}} = \begin{bmatrix} a_{ee} \\ a_{eh} \\ a_{ef} \end{bmatrix}$ such that

$$[i]_{\mathcal{B}}[B]_{\mathcal{B}} \begin{bmatrix} a_{ee} \\ a_{eh} \\ a_{ef} \end{bmatrix} = \begin{cases} 1, & i = e \\ 0, & i \neq e, \end{cases} \qquad \text{where we assume } i \in \{e, f, h\}.$$

Then, as $[e]_{\mathcal{B}} = e_1, [h]_{\mathcal{B}} = e_2, [f]_{\mathcal{B}} = e_3$ we use the above conditions to show that

$$[M_e]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{4} \end{bmatrix},$$

so that

$$M_e = \frac{1}{4}f.$$

Similarly, we find that

$$M_h = \frac{1}{8}h, \; M_f = \frac{1}{4}e.$$

d) Hence,

$$\Omega = \frac{1}{8}(2ef + h^2 + 2fe) = \frac{1}{8}\left(\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\right) = \frac{3}{8}I_2.$$