Math 110, Summer 2012 Long Homework 3 (SOME) SOLUTIONS

Due Tuesday 7/10, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Please write your answers in complete English sentences (where applicable). Make your arguments rigorous - if something is 'obvious', state why this is the case. Full credit will be awarded to those solutions that are complete and answer the question posed in a coherent manner.

1. In this problem you will prove that commuting diagonalisable matrices can be *simultaneously diagonalised.*[1]

Let $f, g \in \mathrm{End}_{\mathbb{C}}(V)$, where V is a finite dimensional $\mathbb{C}$-vector space.

a) Suppose that U is a g-invariant subspace of V. Let $E^g_{\mu_i}$ be the μ_i-eigenspace of g (so that μ_i is an eigenvalue of g). Show that

$$\left(E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k}\right) \cap U = (E^g_{\mu_1} \cap U) \oplus \cdots \oplus (E^g_{\mu_k} \cap U),$$

as follows:

i) Show that
$$E^g_{\mu_1} \cap U + ... + E^g_{\mu_k} \cap U \subset \left(E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k}\right) \cap U.$$

ii) If $W_i = E^g_{\mu_i} \cap U$ show that
$$W_i \cap (\sum_{j \neq i} W_j) = \{0_V\}, \quad \text{for each } i.$$

Hence, we have
$$E^g_{\mu_1} \cap U + ... + E^g_{\mu_k} \cap U = (E^g_{\mu_1} \cap U) \oplus \cdots \oplus (E^g_{\mu_k} \cap U).$$

Suppose that $u \in \left(E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k}\right) \cap U$. Then, $u \in U$ and
$$u = e_1 + ... + e_k,$$
with $e_i \in E^g_{\mu_i}$. You are now going to show that $e_i \in U$, for each i, thereby showing that
$$u \in (E^g_{\mu_1} \cap U) \oplus \cdots \oplus (E^g_{\mu_k} \cap U).$$

Let
$$\Gamma_1 = \{i \in \{1, ..., k\} \mid e_i \in U\},\ \Gamma_2 = \{i \in \{1, ..., k\} \mid e_i \notin U\},$$
so that $\Gamma_1 \cup \Gamma_2 = \{1, ..., k\}$.

iii) Show that if $\Gamma_2 = \varnothing$ then $u \in (E^g_{\mu_1} \cap U) \oplus \cdots \oplus (E^g_{\mu_k} \cap U)$.

iv) Show that if $\Gamma_2 \neq \varnothing$ then
$$u - \sum_{j \in \Gamma_1} e_j \in U.$$

Deduce that if $\Gamma_2 \neq \varnothing$ then there is some nonzero $w \in \left(E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k}\right) \cap U$, such that
$$w = e_{i_1} + ... + e_{i_s}, \quad \text{with } e_{i_j} \in E^g_{\mu_{i_j}} \text{ and } e_{i_j} \notin U.$$

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[1] In fact, if we have a family (A_i) of diagonalisable matrices, such that $A_iA_j = A_jA_i$, for every i, j, then there is a common eigenbasis of all of the A_i: this means there is a *single* matrix P such that
$$P^{-1}A_iP = D_i,$$
with D_i diagonal, for every i.

v) Suppose $\Gamma_2 \neq \varnothing$ and let

$$\mathcal{L} = \{w \in \left(E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k}\right) \cap U \mid w = e_{i_1} + \ldots + e_{i_s}, \text{ with } e_{i_j} \in E^g_{\mu_{i_j}} \text{ and } e_{i_j} \notin U\}.$$

By *iv*) we know that $\mathcal{L} \neq \varnothing$. Let $w \in \mathcal{L}$ with

$$w = e_{i_1} + \ldots + e_{i_s}.$$

Show that it is not possible for $s = 1$. Deduce that we must have $s \geq 2$.

vi) Let $w \in \mathcal{L}$ with

$$w = e_{i_1} + \ldots + e_{i_s},$$

and such that s is minimal. Using v) deduce that there is some $j \in \{1, \ldots, s\}$ such that e_{i_j} is an eigenvector associated to a nonzero eigenvalue μ_{i_j} and show that

$$g(w) - \mu_{i_j} w \in \mathcal{L}.$$

Explain why we have contradicted the minimality condition for w.

vii) Explain why $\Gamma_2 = \varnothing$ and deduce that

$$\left(E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k}\right) \cap U \subset (E^g_{\mu_1} \cap U) \oplus \cdots \oplus (E^g_{\mu_k} \cap U).$$

b) Deduce that if g admits a basis of eigenvectors then there is a basis of U (we are still assuming that U is g-invariant) consisting of eigenvectors of g. (*Hint: Use that* $E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k} = V$ *and* a).)

c) Suppose that $f \circ g = g \circ f$ (we say that f and g *commute*). Let E^f_λ be the λ-eigenspace of f. Prove that E^f_λ is g-invariant. (*Hint: You must show that if* $v \in E^f_\lambda$ *then* $g(v) \in E^f_\lambda$.)

d) Deduce that if f and g commute and there exists a basis of V consisting of eigenvectors of g then there exists a basis of $E^f_{\lambda_i}$ consisting of eigenvectors of g, for every eigenvalue λ_i of f. (*Hint: Use b) and c)*.)

e) Prove: if f and g commute and there exists two bases of V, one consisting of eigenvectors of f and the other consisting of eigenvectors of g, then there is a single basis of V consisting of eigenvectors of both f and g.

f) Prove: Let $A, B \in Mat_n(\mathbb{C})$ such that $AB = BA$. Suppose that A and B are both diagonalisable. Then, there is an invertible matrix P such that

$$P^{-1}AP = D_1, \quad P^{-1}BP = D_2,$$

with D_i a diagonal matrix. (*Hint: This follows from* e).)

g) Find an invertible matrix P such that

$$P^{-1}AP = D_1, \quad P^{-1}BP = D_2,$$

with D_i diagonal, and where

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} -1 & -4 \\ 0 & 3 \end{bmatrix}.$$

(*You do not need to show that $AB = BA$ or that A and B are diagonalisable - although you should be able to see that they are diagonalisable by looking at them.*)

Solution:

a) i) Let $u \in E^g_{\mu_1} \cap U + ... + E^g_{\mu_k} \cap U$. Then,

$$u = e_1 + ... + e_k,$$

where $e_i \in E^g_{\mu_i} \cap U$, by definition of the sum of subspaces. Hence, each $e_i \in U$ since $E^g_{\mu_i} \cap U \subset U$, for each i, so that $u = e_1 + ... + e_k \in U$ (U is a subspace). Moreover,

$$u = e_1 + ... + e_k \in E^g_{\mu_1} + ... + E^g_{\mu_k},$$

as each $e_i \in E^g_{\mu_i} \cap U \subset E^g_{\mu_i}$. By a result obtained in class we have

$$E^g_{\mu_1} + ... + E^g_{\mu_k} = E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k},$$

so that we have $u \in (E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k}) \cap U$.

ii) Let $i \in \{1, \dots, k\}$ and $x \in W_i \cap (\sum_{j \neq i} W_j$. Then, $x \in W_i$ and

$$x = \sum_{j \neq i} y_j \in \sum_{j \neq i} W_j.$$

So, if $x \neq 0_V$ then, for some $j \neq i$, we must have $y_j \neq 0_V$, so that

$$x - \sum_{j \neq i} y_j = 0_V,$$

is a nontrivial linear relation among $\{x\} \cup \{y_j \mid j \neq i\}$. However, since each of the elements in this set are eigenvectors for distinct eigenvalues then this set must be linearly independent so that it is not possible to have a nontrivial linear relation among these vectors. Hence, we must have $x = 0_V$.

iii) Suppose that $\Gamma_2 = \varnothing$. Then, for every $i \in \{1, \dots, k\}$ we have $e_i \in U$ and $e_i \in E^g_{\mu_i}$. Hence, for every i, we have $e_i \in E^g_{\mu_i} \cap U$ so that

$$u = e_1 + ... + e_k \in (E_{\mu_1} \cap U) \oplus \cdots \oplus (E^g_{\mu_k} \cap U).$$

iv) Suppose that $\Gamma_2 \neq \varnothing$. Then, since $u \in U$ and $\sum_{j \in \Gamma_1} e_j \in U$ (because U is a subspace and each $e_j \in U$, when $j \in \Gamma_1$) we have

$$u - \sum_{j \in \Gamma_1} e_j \in U.$$

Hence, if $\Gamma_2 \neq \varnothing$ then there is some nonzero $w \in (E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k}) \cap U$ satisfying the required conditions: namely,

$$w = \sum_{j \in \Gamma_2} e_j.$$

v) Let $w \in \mathcal{L}$ be such that

$$w = e_{i_1} + ... + e_{i_s},$$

with $e_{i_j} \in E^g_{\mu_{i_j}}$ but $e_{i_j} \notin U$. If $s = 1$ then $w = e_{i_1} \in (E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k}) \cap U$ so that $e_{i_1} \in U$. However, by the definition of $\mathcal{L}$ and since $w \in \mathcal{L}$, we can't have $e_{i_1} \in U$. Hence, we must have $s \geq 2$.

vi) Let $w \in \mathcal{L}$ so that

$$w = e_{i_1} + ... + e_{i_s},$$

with $e_{i_j} \in E^g_{\mu_{i_j}}$ and $e_{i_j} \notin U$, for each j. Suppose that s is minimal. By v) we know that $s \geq 2$ so that, since each e_{i_j} is an eigenvector for a distinct eigenvalue, we must have some $j \in \{1, \dots, s\}$ such that e_{i_j} is an eigenvector with nonzero associated eigenvalue (else, $w = e_{i_1} \in \mathcal{L}$ contradicting what we've shown in v)). Now, we see that

$$\begin{aligned}
g(w) - \mu_{i_j} w &= g(e_{i_1} + \dots + e_{i_s}) - \mu_{i_j}(e_{i_1} + \dots + e_{i_s}) \\
&= g(e_{i_1}) + \dots + g(e_{i_s}) - \mu_{i_j}(e_{i_1} + \dots + e_{i_s}) \\
&= \mu_{i_1} e_{i_1} + \dots + \mu_{i_s} e_{i_s} - \mu_{i_j}(e_{i_1} + \dots + e_{i_s}) \\
&= (\mu_{i_1} - \mu_{i_j}) e_{i_1} + \dots + (\mu_{i_{j-1}} - \mu_{i_j}) e_{i_{j-1}} + 0_V + (\mu_{i_{j+1}} - \mu_{i_j}) e_{i_{j+1}} + \dots + (\mu_{i_s} - \mu_{i_j}) e_{i_s}.
\end{aligned}$$

Since $\mu_{i_r} \neq \mu_{i_t}$, for $r \neq t$, we see that $f_l = (\mu_{i_l} - \mu_{i_j}) e_{i_l} \notin U$ (because if $f_l \in U$ then we could scale by $(\mu_{i_l} - \mu_{i_j})^{-1}$ to obtain that $e_{i_l} \in U$, which is absurd as e_{i_l} is assumed ti be not in U). Hence,

$$g(w) - \mu_{i_j} w \in \mathcal{L}.$$

However, $g(w) - w$ is a sum of $s - 1$ vectors, contradicting the minimality condition of w.

vii) Hence, our initial assumption that $\Gamma_2 \neq \varnothing$ must be false, so that $\Gamma_2 = \varnothing$ and, by iii), we must have

$$u \in (E_{\mu_1} \cap U) \oplus \cdots \oplus (E^g_{\mu_k} \cap U).$$

The result follows.

b) If V admits a basis consisting of eigenvectors of g then we must have

$$E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k} = V.$$

Hence, by a), we have

$$U = V \cap U = (E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k}) \cap U = (E_{\mu_1} \cap U) \oplus \cdots \oplus (E^g_{\mu_k} \cap U).$$

Thus, we find a basis $\mathcal{B}$ of U by finding a basis $\mathcal{B}_i$ (possibly empty) of each $E^g_{\mu_i} \cap U$ and since $E^g_{\mu_i} \cap U \subset E^g_{\mu_i}$ we have that $\mathcal{B}_i$ consists of eigenvectors of g. Hence, there is a basis of U consisting of eigenvectors of g.

c) Let $v \in E^f_\lambda$, we want to show that $g(v) \in E^f_\lambda$, so that, if $g(v) \neq 0_V$, then $g(v)$ is an eigenvector of f with associated eigenvalue λ. Now,

$$f(g(v)) = g(f(v)) = g(\lambda v) = \lambda g(v),$$

so that $g(v) \in E^f_\lambda = \{w \in V \mid f(w) = \lambda w\}$.

d) By c) we see that, for every λ_i, we have $E^f_{\lambda_i}$ is g-invariant. Hence, by b) since there is a basis of V consisting of eigenvectors of g then we can find a basis of $E^f_{\lambda_i}$ consisting of eigenvectors of g, for every λ_i.

e) We just need to combine d) and the assumption that there is a basis of V consisting of eigenvectors of f. In this case, we must have

$$E^f_{\lambda_1} \oplus \cdots \oplus E^f_{\lambda_l} = V.$$

By d) we can find a basis $\mathcal{C}_i$ of each $E^f_{\lambda_i}$ consisting of eigenvectors of g. Since $\mathcal{C}_i \subset E^f_{\lambda_i}$ the vectors in $\mathcal{C}_i$ are also eigenvectors of f. Since the above sum is direct, we have that

$$\mathcal{C} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_l,$$

is a basis of V. Moreover, $\mathcal{C}$ consists of eigenvectors of both f and g.

f) The assumptions on A, B allow us to apply e) to the morphisms $T_A, T_B \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$. We see that there is a simultaneous eigenbasis $\mathcal{C}$ of both T_A, T_B. If $\mathcal{C} = (c_1, \dots, c_n)$ then denoting

$$P = [c_1 \cdots c_n] \in GL_n(\mathbb{C}),$$

we have that

$$P^{-1}AP = P_{\mathcal{C}\leftarrow\mathcal{S}^{(n)}}[T_A]_{\mathcal{S}^{(n)}}P_{\mathcal{S}^{(n)}\leftarrow\mathcal{C}} = [T_A]_{\mathcal{C}} = D_1,$$

and

$$P^{-1}BP = P_{\mathcal{C}\leftarrow\mathcal{S}^{(n)}}[T_B]_{\mathcal{S}^{(n)}}P_{\mathcal{S}^{(n)}\leftarrow\mathcal{C}} = [T_B]_{\mathcal{C}} = D_2,$$

with both D_1, D_2 diagonal.

g) The above theory tells us that in order to determine a simultaneous eigenbasis for A and B we must proceed as follows: each eigenspace E^A_λ of A is B-invariant so that B defines an endomorphism of E^A_λ (by c) above). We then have that this endomorphism is diagonalisable (this is d) above) so we need can find an eigenbasis for this endomorphism of E^A_λ. Then, we can use these eigenbases for each eigenspace of A to determine a simultaneous eigenbasis for both A and B.

We have that

$$\chi_A(\lambda) = (2-\lambda)(1-\lambda),$$

so that $\lambda_1 = 1, \lambda_2 = 2$ are the eigenvalues of A. Then, it is easy to see that

$$E_{\lambda_1} = \mathrm{span}_{\mathbb{C}}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\},\ E_{\lambda_2} = \left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}.$$

Thus, since each eigenspace of A has dimension 1 and is B-invariant, we must have that

$$B\begin{bmatrix}1\\-1\end{bmatrix} \in E_{\lambda_1},\ B\begin{bmatrix}1\\0\end{bmatrix} \in E_{\lambda_2}.$$

This is also easy to verify directly. Hence, a simultaneous eigenbasis of A and B is

$$\left(\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right),$$

so that if we take

$$P = \begin{bmatrix}1 & 1\\-1 & 0\end{bmatrix},$$

then we have

$$P^{-1}AP = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix},\ P^{-1}BP = \begin{bmatrix}3 & 0\\0 & -1\end{bmatrix}.$$