# **Math 110, Summer 2012** Long Homework 2 (SOME) SOLUTIONS

*Due Tuesday 7/3, 10.10am, in Etcheverry 3109. Late homework will not be accepted.*

*Please write your answers in complete English sentences (where applicable). Make your arguments rigorous - if something is 'obvious', state why this is the case. Full credit will be awarded to those solutions that are complete and answer the question posed in a coherent manner.*

1. Let $V$ be a $\mathbb{K}$-vector space. Consider the following *minimal spanning* property of a spanning set $E \subset V$ (so that $\text{span}_{\mathbb{K}} E = V$):

   $*$ if $E' \subset E$ and $\text{span}_{\mathbb{K}} E = V$ then $E' = E$.

Prove that a subset $E \subset V$ that spans $V$, so that $\text{span}_{\mathbb{K}} E = V$, and which satisfies the minimal spanning property $*$ is a basis of $V$.

(*This is Proposition 1.5.9 on p.32 of the notes. To show that $E$ is a basis of $V$ it suffices to show that $E$ is linearly independent. Look at the proof of Proposition 1.5.5 to help you show how you can use the minimal spanning property of $E$ to obtain linear independence of $E$.*)

*Solution:* We are going to show that a spanning set $E$ that satisfies $*$ is linearly independent: suppose that $E$ were linearly dependent (ie, NOT linearly indepdendent). Then, by the Elimination Lemma there exists some $v \in E$ such that

$$\text{span}_{\mathbb{K}} E \setminus \{v\} = \text{span}_{\mathbb{K}} E = V.$$

Therefore, if $E' = E \setminus \{v\}$ then we have $E' \subset E$ and

$$\text{span}_{\mathbb{K}} E' = V.$$

Hence, by the minimal spanning property $*$, we must have that $E' = E$, which is clearly absurd since $v \in E$ while $v \notin E'$. Therefore, our initial assumption that $E$ is not linearly dependent cannot be true so that $E$ must be linearly independent. Hence, $E$ is a spanning linearly independent subset of $V$ so must be a basis.

2. In this problem we are going to try and determine properties of a matrix $A \in Mat_n(\mathbb{K})$ by studying endomorphisms of $Mat_n(\mathbb{K})$.

Let $A \in Mat_n(\mathbb{K})$. Define the linear morphisms (you DO NOT have to show this)

$$L_A : Mat_n(\mathbb{K}) \to Mat_n(\mathbb{K})\ ;\ B \mapsto AB, \quad R_A : Mat_n(\mathbb{K}) \to Mat_n(\mathbb{K})\ ;\ B \mapsto BA.$$

a) Prove that $L_A$ is injective if and only if $A$ is invertible.

b) Prove that $R_A$ is injective if and only if $A$ is invertible.

c) Deduce that $L_A$ is injective if and only if $R_A$ is injective.

   (*Theorem 1.7.4 might be useful for parts a), b).*)

Now, consider

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \in Mat_2(\mathbb{Q}),$$

and let $\mathcal{B} = (e_{11}, e_{12}, e_{21}, e_{22})$ denote the standard ordered basis of $Mat_2(\mathbb{Q})(= \mathbb{K}^{[2]\times[2]})$.

We know, using determinants for example, that $A$ is invertible. However, we are going to obtain this fact using the results you have just proved above.

d) Determine the matrix of $L_A$ relative to $\mathcal{B}$, $[L_A]_{\mathcal{B}} \in Mat_4(\mathbb{Q})$.

e) Show that $L_A$ is injective. (*Use Theorem 1.7.4*)

f) Deduce that $A$ is invertible.

g) By solving the matrix equation

$$[L_A]_{\mathcal{B}}\underline{x} = [I_2]_{\mathcal{B}},$$

find the inverse of $A$.

*Solution:*

a) If $L_A$ is injective then $L_A$ is surjective, since $L_A$ is a linear endomorphism of a finite dimensional vector space (Theorem 1.7.4). In particular, there exists $B \in Mat_n(\mathbb{K})$ such that

$$L_A(B) = I_n \implies AB = I_n,$$

so that $A$ is invertible (you should already know that a left inverse of $A$ is also a right inverse of $A$. This was also proved last Thursday in class). Conversely, suppose that $A^{-1}$ exists. Then, if

$$AB = 0_n \implies A^{-1}(AB) = A^{-1}0_n = 0_n \implies B = 0_n.$$

Hence, if $B \in \ker L_A$ then $B = 0_n$ so that $L_A$ is injective.

b) This is the same as *a*).

c) Combine *a*) and *b*) to get the result.

d) We have

$$[L_A]_{\mathcal{B}} = [[L_A(e_{11})]_{\mathcal{B}}[L_A(e_{12})]_{\mathcal{B}}[L_A(e_{21})]_{\mathcal{B}}[L_A(e_{22})]_{\mathcal{B}}] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

e) Since there is a pivot in every column we see that $L_A$ is injective, by Theorem 1.7.4.

f) By *a*) we know that $A$ is invertible since $L_A$ is injective.

g) We consider the augmented matrix

$$[[L_A]_{\mathcal{B}}\ [I_2]_{\mathcal{B}}] = \begin{bmatrix} 1 & 0 & -1 & 0 & |1 \\ 0 & 1 & 0 & -1 & |0 \\ 0 & 0 & 1 & 0 & |0 \\ 0 & 0 & 0 & 1 & |1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & |1 \\ 0 & 1 & 0 & 0 & |1 \\ 0 & 0 & 1 & 0 & |0 \\ 0 & 0 & 0 & 1 & |1 \end{bmatrix},$$

so that the solution is

$$\underline{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

This is the $\mathcal{B}$-coordinate vector of the inverse of $A$: hence, we have

$$A^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$