Math 110, Summer 2012 Long Homework 1, SOLUTIONS

Due Tuesday 6/26, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Please write your answers in complete English sentences (where applicable). Make your arguments rigorous - if something is 'obvious', state why this is the case. Full credit will be awarded to those solutions that are complete and answer the question posed in a coherent manner.

1. Let V be a $\mathbb{K}$-vector space, $E \subset V$ a nonempty finite subset. In this question we will characterise properties of E using linear morphisms.

a) Prove: E is linearly independent if and only if the linear morphism

$$h : \mathbb{K}^E \to V \; ; \; f \mapsto \sum_{e \in E} f(e) \cdot e,$$

is injective.

b) Prove: E is a spanning set of V if and only if the linear morphism

$$h : \mathbb{K}^E \to V \; ; \; f \mapsto \sum_{e \in E} f(e) \cdot e,$$

is surjective.

Solution: *a*) ($\Rightarrow$) Suppose that E is linearly independent. Denote $E = \{e_1, \dots, e_n\}$, where $n = |E|$. Then, we want to show that the linear morphism h is injective. Using a result from the notes, we know that h is injective if and only if $\ker h = \{0_{\mathbb{K}^E}\}$. Let $f \in \ker h$: we will show that $f = 0_{\mathbb{K}^E}$. As $h(f) = 0_V$, we have

$$0_V = h(f) = \sum_{e \in E} f(e)e = f(e_1)e_1 + \dots + f(e_n)e_n, \quad f(e_1), \dots, f(e_n) \in \mathbb{K}.$$

This is a linear relation among the vectors in E so that it must be the trivial linear relation, since E is linearly independent. Hence, $f(e_1) = \dots = f(e_n) = 0 \in \mathbb{K}$. This means that f is the zero function, ie, $f = 0_{\mathbb{K}^E}$.

($\Leftarrow$) Suppose that h is injective. Then, $\ker h = \{0_{\mathbb{K}^E}\}$. Let

$$\lambda_1 e_1 + \dots + \lambda_n e_n = 0_V,$$

be a linear relation among the vectors in E. Then, defined the function $f \in \mathbb{K}^E$ as follows

$$f : E \to \mathbb{K} \; ; \; e_i \mapsto \lambda_i.$$

Then, the linear relation above can be translated as saying that $h(f) = 0_V$. Hence, $f = 0_{\mathbb{K}^E}$, so that $\lambda_i = f(e_i) = 0$, for every i. Therefore, E is linearly independent.

b) ($\Rightarrow$) Suppose that $\text{span}_{\mathbb{K}} E = V$. Thus, for every $v \in V$, there are scalars $c_1, \dots, c_n \in \mathbb{K}$ such that

$$v = c_1 e_1 + \dots + c_n e_n.$$

We want to show that h is surjective: therefore, for any $v \in V$ we must find a function $f \in \mathbb{K}^E$ such that

$$h(f) = f(e_1)e_1 + \dots + f(e_n)e_n = v.$$

So, let $v \in V$ and $c_1, \dots, c_n$ be scalars as above. Define $f \in \mathbb{K}^E$ to be the function such that $f(e_i) = c_i$, for any i. Then, we have $h(f) = v$. Hence, h is surjective.

($\Leftarrow$) Conversely, suppose that h is surjective. Hence, for any $v \in V$ there is some function $f \in \mathbb{K}^E$ such that $h(f) = v$. In order to show that $\text{span}_{\mathbb{K}} E = V$ we must show that every $v \in V$ can be written as a linear combination of elements in E. So, let $v \in V$. Then, since h is surjective, we can find $f \in \mathbb{K}^E$ such that

$$v = h(f) = f(e_1)e_1 + \dots + f(e_n)e_n,$$

with $f(e_i) \in \mathbb{K}$. Hence, we have written v as a linear combination of vectors in E. Since v was arbitrary this shows that $\text{span}_{\mathbb{K}} E = V$.

2. Consider the subspace

$$sl_2(\mathbb{C}) = \left\{ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in Mat_2(\mathbb{C}) \mid a_{11} + a_{22} = 0 \right\} \subset Mat_2(\mathbb{C}).$$

So, $sl_2(\mathbb{C}) = \ker \text{tr}$, where tr is the linear morphism (you DO NOT have to show this)

$$\text{tr} : Mat_2(\mathbb{C}) \to \mathbb{C} \; ; \; A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mapsto a_{11} + a_{22}.$$

This is the *special linear Lie (pronounced 'lee') algebra of* 2×2 *complex matrices*. It is of fundamental importance and arises in many areas of mathematics. We denote

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in sl_2(\mathbb{C}).$$

a) Using the Rank Theorem, show that $\dim sl_2(\mathbb{C}) = 3$.

b) Show that $\mathcal{B} = (e, h, f)$ is an ordered basis of $sl_2(\mathbb{C})$.

For every $A \in M_2(\mathbb{C})$ we have a function

$$\text{ad}_A : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \; ; \; B \mapsto AB - BA.$$

c) Show that ad_A is a linear morphism, for every $A \in M_2(\mathbb{C})$.

d) Let $A, B \in sl_2(\mathbb{C})$. Show that $\text{ad}_A(B) \in sl_2(\mathbb{C})$.

Hence, for $A \in sl_2(\mathbb{C})$ we see that $\text{ad}_A \in \text{End}_{\mathbb{C}}(sl_2(\mathbb{C}))$ so that there exists a function

$$\text{ad} : sl_2(\mathbb{C}) \to \text{End}_{\mathbb{C}}(sl_2(\mathbb{C})) \; ; \; A \mapsto \text{ad}_A.$$

e) Determine $[\text{ad}_e]_{\mathcal{B}}, [\text{ad}_h]_{\mathcal{B}}, [\text{ad}_f]_{\mathcal{B}}$, the matrices of $\text{ad}_e, \text{ad}_h, \text{ad}_f$ with respect to the ordered basis $\mathcal{B}$.

f) The function ad is linear (you DO NOT have to show this): hence, if $A = \lambda e + \mu h + \tau f \in sl_2(\mathbb{C})$, $\lambda, \mu, \tau \in \mathbb{C}$, then $\text{ad}_A = \lambda \text{ad}_e + \mu \text{ad}_h + \tau \text{ad}_f$. Show that ad is injective.

Solution: a) The Rank Theorem states that, if $f \in \text{Hom}_{\mathbb{K}}(V, W)$ then

$$\dim V = \dim \ker f + \dim \text{im} f.$$

So, since $sl_2(\mathbb{C}) = \ker \text{tr}$, where $\text{tr} : Mat_2(\mathbb{C}) \to \mathbb{C}$, if we can show that $\dim \text{imtr} = 1$ then we are done. As $\text{imtr} \subset \mathbb{C}$ is a subspace, we must have that its dimension is either 0 or 1. If $\dim \text{imtr} = 0$ then $\text{imtr} = \{0\}$ so that tr is the zero morphism. However, we have

$$\text{tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = 1,$$

so that tr is not the zero morphism. Hence, we must have that $\text{imtr} = \mathbb{C}$ and $\dim \text{imtr} = 1$. Hence, by the Rank Theorem we have

$$4 = 1 + \dim sl_2(\mathbb{C}),$$

and the claim follows.

b) Since $\dim sl_2(\mathbb{C}) = 3$ it suffices to show that $\mathcal{B}$ is linearly independent (because $|\mathcal{B}| = 3$) in order to show that it is a basis. So, suppose we have a linear relation

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c_2 & c_1 \\ c_3 & -c_2 \end{bmatrix}.$$

Then, we must have $c_1 = c_2 = c_3 = 0$ so that $\mathcal{B}$ is linearly independent.

c) Fix $A \in Mat_2(\mathbb{C})$. We show that ad_A satisfies LIN: let $B, C \in Mat_2(\mathbb{C}), \lambda \in \mathbb{C}$. Then,

$$\text{ad}_A(B + \lambda C) = A(B + \lambda C) - (B + \lambda C)A = AB - BA + A\lambda C - \lambda CA = \text{ad}_A(B) + \lambda \text{ad}_A(C).$$

Hence, ad_A is linear.

d) Let

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \ B = \begin{bmatrix} x & y \\ z & -x \end{bmatrix} \in sl_2(\mathbb{C}).$$

Then,

$$AB - BA = \begin{bmatrix} xa + zb & ya - xb \\ xc - za & yc + xa \end{bmatrix} - \begin{bmatrix} ax + cy & bx - ay \\ az - cx & bz + ax \end{bmatrix},$$

and it is easy to see that this matrix has zero trace.

e) We have

$$[\text{ad}_e]_{\mathcal{B}} = [[\text{ad}_e(e)]_{\mathcal{B}} \ [\text{ad}_e(h)]_{\mathcal{B}} \ [\text{ad}_e(f)]_{\mathcal{B}}],$$

and

$$\text{ad}_e(e) = e.e - e.e = 0, \ \text{ad}_e(h) = eh - he = -2e, \ \text{ad}_e(f) = ef - fe = h,$$

so that

$$[\text{ad}_e]_{\mathcal{B}} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similarly,

$$[\text{ad}_h]_{\mathcal{B}} = [[\text{ad}_h(e)]_{\mathcal{B}} \ [\text{ad}_h(h)]_{\mathcal{B}} \ [\text{ad}_h(f)]_{\mathcal{B}}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

$$[\text{ad}_f]_{\mathcal{B}} = [[\text{ad}_f(e)]_{\mathcal{B}} \ [\text{ad}_f(h)]_{\mathcal{B}} \ [\text{ad}_f(f)]_{\mathcal{B}}] = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

f) We are going to show that $\ker \text{ad} = \{0\}$. Suppose that $\text{ad}_A = 0$. Then, since $\mathcal{B}$ is a basis we can write

$$A = c_e e + c_h h + c_f f,$$

for some unique scalars $c_e, c_h, c_f \in \mathbb{C}$. Hence, using the linearity of ad, we have

$$0 = \text{ad}_A = c_e \text{ad}_e + c_h \text{ad}_h + c_f \text{ad}_f,$$

so that if we can show that $E = \{\text{ad}_e, \text{ad}_h, \text{ad}_f\}$ are linearly independent then we obtain $c_e = c_h = c_f = 0 \in \mathbb{C}$ so that $A = 0$. But linear independence of E is the same as showing that the matrices $[\text{ad}_e]_{\mathcal{B}}, [\text{ad}_h]_{\mathcal{B}}, [\text{ad}_f]_{\mathcal{B}}$ are linearly independent, which is easy to verify by looking at them. Hence, we have that E is linearly independent so that $A = 0$ and ad is injective.