Math 110, Spring 2014. Quotient Spaces Review

Let V be a vector space over F (recall that we always assume that $F \in \{\mathbb{R}, \mathbb{C}\}$) and $U \subset V$ be a subspace.

We define an **affine subset of V parallel to U** to be a subset of the form

$$v + U \stackrel{def}{=} \{v + u \mid u \in U\}$$

for some $v \in V$. We could also say that the above subset is an **affine subset of V parallel to U through v**.

It is important to note that an affine subset parallel to U is **not** a subspace of U in general, but only a subset (ie, it doesn't necessarily satisfy the axioms defining a subspace). The only affine subset parallel to U that is a subspace is the subset $0_V + U = U$.

Important Remark: when we consider an affine subset parallel to U as above we have defined it with respect to some $v \in V$, ie, we have implicitly referred to some $v \in V$. However, this v is **not** unique! In fact, infinitely many different v's will define the **same** affine subset parallel to U: for any $w \in U$ we have that

$$v + U = (v + w) + U.$$

Indeed, we need only show that the two subsets above are equal: let $v + u \in v + U$ (recall the definition of the set $v + U$). Then,

$$v + u = (v + w) + (u - w) \in (v + w) + U,$$

since $u - w \in U$. Hence, $v + U \subset (v + w) + U$.

The other containment is similar. We will call such a v defining an affine subset parallel to U a **representative** of the affine subset.

Hence, there is a certain ambiguity inherent in our definition of an affine subset parallel to U - when writing down such a subset we make reference to some $v \in V$, but there is no natural choice of v to take. However, this is not such a problem: let's say that I have an affine subset parallel to U in my pocket, let's denote it $v + U$, and let's say that you have an affine subset paralle to U in your pocket, denoted $v' + U$. How can we tell when they are the **same** affine subset? We have the following nice

FACT 1:

$$v + U = v' + U \iff v - v' \in U \iff v' - v \in U.$$

Let's look at some (geometric) examples:

i) let $V = \mathbb{R}^2$, and $U \subset \mathbb{R}^2$ be some line (ie $\dim U = 1$). Then, an affine subset parallel to U is a **line parallel to** U in the usual sense; ie, it's a line through some point in $\mathbb{R}^2$ in the direction of U.

ii) let $V = \mathbb{R}^3$ and U be a line (so that $\dim U = 1$). Then, an affine subset parallel to U is a line in $\mathbb{R}^3$ parallel to U in the usual sense; ie, it's a line through some point in $\mathbb{R}^3$ in the same direction as U.

If $\dim U = 2$ (so that U is a plane in $\mathbb{R}^3$) then an affine subset parallel to U is a plane in $\mathbb{R}^3$ that is parallel to U in the usual sense.

Basically, when I think of an affine subset parallel to some subspace U in an arbitrary V I have the above two examples in my head.

Now, let's make things a bit more interesting: denote the **set of all affine subsets parallel to U by V/U** so that

$$V/U \overset{def}{=} \{S \subset V \mid S = v + U, \text{ for some } v \in V\}.$$

We can define the structure of a vector space (over F) on V/U:

- we have $0_{V/U} = 0_V + U \in V/U$,
- for $v + U, w + U \in V/U$ we define

$$(v + U) + (w + U) \overset{def}{=} (v + w) + U.$$

- for $v + U \in V/U$, $c \in F$ we define

$$c(v + U) \overset{def}{=} (cv) + U.$$

FACT 2: with the above definitions V/U is a vector space over F.

So, now we can ask questions about linear independence, span, bases, whatever, in V/U.

Example: Let $U = \{\underline{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\} \subset \mathbb{R}^3 (= V)$. Prove that the list $\left(\begin{bmatrix}1\\1\\1\end{bmatrix} + U, \begin{bmatrix}1\\-1\\1\end{bmatrix} + U\right)$ is linearly dependent in V/U.

We need to find $c_1, c_2 \in F$, with at least one being nonzero, such that

$$c_1\left(\begin{bmatrix}1\\1\\1\end{bmatrix} + U\right) + c_2\left(\begin{bmatrix}1\\-1\\1\end{bmatrix} + U\right) = 0_{V/U}.$$

Using the definitions of 'sum', 'scalar multiplication' and 'zero vector' given above, the above equation is the same as

$$\left(c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\\1\end{bmatrix}\right) + U = 0_V + U.$$

Hence, we are saying the the affine subset on the LHS is equal to the affine subset on the RHS - using FACT 1 we must have

$$c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}c_1 + c_2\\c_1 - c_2\\c_1 + c_2\end{bmatrix} \in U.$$

Thus, we necessarily must have that (by definition of U)

$$0 = (c_1 + c_2) + (c_1 - c_2) + (c_1 + c_2) = 3c_1 + c_2.$$

Taking $c_1 = 1, c_2 = -3$ the above equation is satisfied so that we have found a nontrivial linear relation among $\left(\begin{bmatrix}1\\1\\1\end{bmatrix} + U, \begin{bmatrix}1\\-1\\1\end{bmatrix} + U\right)$. Hence, this list is linearly dependent.

We could have proved the above statement quite easily using the

FACT 3: let V be finite dimensional, $U \subset V$ a subspace. Then,

$$\dim V = \dim U + \dim V/U.$$

Example: (*This is half of the proof of FACT 3*) Let V be finite dimensional, $U \subset V$ a subspace. We know that we can find a subspace $W \subset V$ such that $V = U \oplus W$. Let $(w_1, \ldots, w_k) \subset W$ be a basis of W. Prove that

$$(w_1 + U, \ldots, w_k + U)$$

is linearly independent in V/U.

This is a linear independence statement: suppose that $c_1, \ldots, c_k \in F$ such that

$$c_1(w_1 + U) + \ldots + c_k(w_k + U) = 0_{V/U}.$$

Thus, we must have

$$(c_1 w_1 + \ldots + c_k w_k) + U = 0_V + U,$$

so that

$$c_1 w_1 + \ldots + c_k w_k \in U.$$

Notice that the LHS is a vector in W, so we have shown that

$$c_1 w_1 + \ldots + c_k w_k \in U \cap W = \{0_V\},$$

since $V = U \oplus W$. Thus, we have

$$c_1 w_1 + \ldots + c_k w_k = 0_V,$$

so that $c_1 = \cdots = c_k = 0$, since the w's are linearly independent.

In fact, the above example gives us a way to try and find a basis of V/U (at least for V finite dimensional): *take U, find a subspace W such that $V = U \oplus W$. Let $(w_1, \ldots, w_k)$ be a basis of W. Then, the list $(w_1 + U, \ldots, w_k + U)$ is a basis of V/U.*

Note: it is not enough to merely take linearly independent $(w_1, \ldots, w_k) \subset V$ with $w_1, \ldots, w_k \notin U$ - for example, if $V = \mathbb{R}^3$ and $U = \text{span}(e_1)$, then $w_1 = e_2, w_2 = e_1 + e_2$ are vectors not in U and are linearly independent. However, since

$$w_2 - w_1 = e_1 \in U,$$

we see that $w_1 + U = w_2 + U$ so that the list $(w_1 + U, w_2 + U)$ is linearly dependent!

Example: Let V be finite dimensional, $U \subset V$ a subspace. Suppose that $W \subset V$ is a subspace such that $v = U \oplus W$. Construct an explicit isomorphism

$$T : V/U \to W.$$

This problem is asking that we construct a linear map

$$T : V/U \to W,$$

that is bijective. However, since $\dim V/U = \dim V - \dim U = \dim W$, and both these vector spaces are finite dimensional, we need only show that T is injective or surjective (why?). How do we construct linear maps? We need only define it on a basis of V/U - so, choose a basis

$(w_1, \ldots, w_k)$ of W, so that (by the above Example) $(w_1 + U, \ldots, w_k + U)$ is a basis of V/U. What do we think the outputs of these basis vectors should be? An obvious choice would be

$$T(w_i + U) = w_i, \ i = 1, \ldots, k.$$

Let's show that T with this definition is injective: suppose that $v + U \in \text{null}(T)$. Thus, we have

$$v + U = \sum_{i=1}^{k} c_i(w_i + U) = (\sum_{i=1}^{k} c_i w_i) + U,$$

so that (recalling the definition of a linear map defined by specifying the outputs of a basis - Theorem 3.5)

$$0 = T(v + U) = \sum_{i=1}^{k} c_i w_i \implies c_1 = \cdots = c_k = 0,$$

since the w's are linearly independent. Hence, $v + U = 0_V + U = 0_{V/U}$ and $\text{null}(T) = \{0\}$ so that T is injective.