# **Math 110, Spring 2014.** Solutions to HW1, 1.C Q8, Q12

8) EDIT: an easier example was outlined to me. I leabe my (very) involved example below. Hopefully it's of some use to someone...

Take $f(x) = x$ to be periodic with period 1 (so that $f(0) = f(1) = 0$), and $g(x) = x$ to be periodic with period $\sqrt{2}$ (so that $g(0) = g(\sqrt{2}) = 0$. Then, $f + g$ is not periodic - if it were, say of period $r$, then we would have $(f + g)(r) = (f + g)(0) = f(0) + g(0) = 0$. This implies that $f(r) = g(r) = 0$, since $f(x) \geq 0$ and $g(x) \geq 0$, for every $x \in \mathbb{R}$. Thus, we must have that $r$ is an integer (since $f(r) = 0$) and that $r$ is an integer multiple of $\sqrt{2}$. - say, $d = r = c\sqrt{2}$, for $c, d \in \mathbb{Z}$. Hence, we would have $\sqrt{2} \in \mathbb{Q}$, which is absurd.

*(Involved example)* Consider the periodic functions

$$f : \mathbb{R} \to \mathbb{R} \; ; \; x \mapsto \begin{cases} 1, \; x \in [2p, 2p+1], p \in \mathbb{Z}, \\ 0, x \in (2q+1, 2(q+1)), q \in \mathbb{Z}, \end{cases}$$

$$g : \mathbb{R} \to \mathbb{R} \; ; \; x \mapsto \begin{cases} 0, \; x \in [2\sqrt{2}p, 2\sqrt{2}p + \sqrt{2}], p \in \mathbb{Z} \\ 1, x \in (2\sqrt{2}q + \sqrt{2}, 2\sqrt{2}(q+1)), q \in \mathbb{Z}. \end{cases}$$

So that $f$ has period 2, $g$ has period $2\sqrt{2}$, and they are both 'step functions' (draw the graph of these functions when $x \in (-10, 10)$, say).

Suppose that $(f + g)$ has period $r$. Then, in particular, for $x \in [0, 1]$, we must have that

$$(f + g)(x + r) = (f + g)(x) = f(x) + g(x) = 1 + 0 = 1.$$

We claim that this is impossible: namely, we prove that if $s \geq 0$ is such that

$$(f + g)(x + s) = 1, \quad \text{for } x \in [0, 1],$$

then $s = 0$. Indeed, suppose that $s \neq 0$ and the above holds. Then, the interval $[s, s + 1]$ must intersect a unique interval $J = [2p, 2p + 1]$, for some (unique) $p \in \mathbb{Z}$ (think of dividing up the number line into successive closed and open intervals of length one). Hence, since for any $x \in J \cap [s, s + 1]$ we have $f(x) = 1$ (dy definition of $f$), we must have $g(x) = 0$, for every $x \in J \cap [s, s + 1]$ (since $1 = f(x) + g(x) = (f + g)(x) = (f + g)(x - s)$).

Suppose that $J \neq [s, s+1]$. Moreover, assume that $2p \in [s, s+1]$ (ie the unit interval $[s, s+1]$ intersects the 'left side' of $J$ - draw a picture!). Since we have assumed that $J \neq [s, s + 1]$, we have

$$[s, s + 1] = [s, 2p) \cup [2p, s + 1],$$

and both intervals are nonempty. Now, for $x \in [s, 2p) \subset (2p-1, 2p)$ we have $f(x) = 0$, so that $g(x) = 1$, and we've seen above that for $x \in [2p, s + 1] = J \cap [s, s + 1]$, we have $f(x) = 1$ and $g(x) = 0$. Hence, $2p$ must be a point where $f$ and $g$ simultaneously change values from 0 to 1 and vice versa. Hence, since we must have that $2p = 2\sqrt{2}q$, for some $q \in \mathbb{Z}$, which implies that $\sqrt{2} \in \mathbb{Q}$, which is absurd. We obtain a similar absurdity if we assume that $2p + 1 \in [s, s + 1]$. Hence, we must have that $J = [s, s + 1]$ (ie, $s = 2p$, for some $p \in \mathbb{Z}$).

Since we have assumed that $s > 0$, then we have $p > 0$. Moreover, we've shown that $g(x) = 0$, for every $x \in [s, s + 1]$. Hence, $[s, s + 1] \subset [2\sqrt{2}q, 2\sqrt{2}q + +\sqrt{2}]$, for some $q$ (by definition of $g$). It is then possible to show (I won't do it as this is getting pretty tedious!) that the left end points of $[s, s + 1]$ and $[2\sqrt{2}q, 2\sqrt{2}q + \sqrt{2}]$, would have to agree (by considering that $(f + g)(x + s) = (f + g)(x) = 1$, for $x \in (1, \sqrt{2})$, and with $f(x) = 0$, $g(x) = 1$). Hence, we would obtain that $2p = s = 2\sqrt{2}q$, so that $\sqrt{2} \in \mathbb{Q}$, (here we'd use that $p > 0, q > 0$) which is again a contradiction. Hence, we must have $s = 0$...! But if $s = 0$, then it is not possible that $(f+g)(x+r) = 1$, for $x \in [0,1]$, so that $(f+g)$ is not periodic of period $r$.

12) Let $U_1, U_2, U_3 \subset V$. Assume that $U_1 \cup U_2 \cup U_3$ is a subspace of $V$. We split into two cases:

1) $U_1 \cup U_2$ is a subspace of $V$ - then, by Q11, we have that $U_1 \subset U_2$ or $U_2 \subset U_1$. If $U_1 \subset U_2$, say, then we are going to show that either $U_1 \subset U_3$ and $U_2 \subset U_3$, or that $U_1 \subset U_2$ and $U_3 \subset U_2$. Indeed, if we let $W = U_1 \cup U_2 (= U_2)$, then by applying Q11 again to $W \cup U_3$ (which is a subspace), we must have $W \subset U_3$ or $U_3 \subset W$. That is, either $U_1 \subset U_2 \subset U_3$, or $U_3 \subset U_2$ and $U_1 \subset U_2$. If we assume that $U_2 \subset U_1$ we can proceed similarly.

2) Suppose that $U_1 \cup U_2$ is not a subspace, so that $U_2 \not\subset U_1$ and $U_1 \not\subset U_2$ (by Q11). Now, if $u_1 \in U_1, u_1 \notin U_2$, then for any $0 \neq c \in F$ we have $cu_1 \in U_1, cu_1 \notin U_2$ (can you see why?). Similarly, for any $u_2 \in U_2, u_2 \notin U_1$, we have that for any $0 \neq d \in F$, $du_2 \in U_2, du_2 \notin U_1$. Now, $u_1 + u_2 \in U_1 \cup U_2 \cup U_3$, since we've assumed it's a subspace. Hence, we must have that either

   - $u_1 + u_2 \in U_1$ - which implies that $u_2 \in U_1$, which is absurd;
   - $u_1 + u_2 \in U_2$ - which implies that $u_1 \in U_2$, which is absurd;
   - $u_1 + u_2 \in U_3$.

   Similarly, we can show that $2u_1 + u_2 \in U_3$. Hence, we have

   $$u_1 = (2u_1 + u_2) - (u_1 + u_2) \in U_3.$$

   A similar argument shows that, for any $u_2 \in U_2, u_2 \notin U_1$, we have $u_2 \in U_3$. Hence, if we define

   $$A = \{u_1 \in U_1 \mid u_1 \notin U_2\} \neq \varnothing, \ B = \{u_2 \in U_2 \mid u_2 \notin U_1\} \neq \varnothing,$$

   then we have $A \subset U_3, B \subset U_3$. Notice that

   $$U_1 \cup U_2 = A \cup B \cup (U_1 \cap U_2),$$

   (think of Venn diagrams) so if we can show that $U_1 \cap U_2 \subset U_3$ then we are done!

   Suppose that $U_1 \cap U_2 \not\subset U_3$, so that there is some $0 \neq w \in U_1 \cap U_2, w \notin U_3$. Consider $u_1 + w + u_2$, where $u_1 \in A, u_2 \in B$. Then, we have that $u_1 + u_2 + u_3 \in U_1 \cup U_2 \cup U_3$, and we must have either

   - $u_1 + w + u_2 \in U_1$, which implies that $u_2 \in U_1$, since $(u_1 + w) \in U_1$. But this can't be true, since $u_2 \in B$.
   - $u_1 + w + u_2 \in U_2$, which implies that $u_1 \in U_2$, which is again a contradiction.
   - So we must have that $u_1 + w + u_2 \in U_3$, but then this implies that $w \in U_3$, since $u_1, u_2 \in U_3$ and so $u_1 + u_2 \in U_3$.

   Since none of these outcomes is possible we have made a false assumption in supposing that $U_1 \cap U_2 \not\subset U_3$ (ie, we can't find such a $w$ as above).

The result follows.