Math 110, Spring 2014. Dual Spaces Review

Let V be a vector space (over F). Recall that the **dual space of V, denoted V',** is the set of all linear maps from V to the one dimensional vector space F:

$$V' = \{T : V \to F \mid T \text{ is a linear map}\}.$$

Also, in Axler's notation, we write $V' = L(V, F)$.

Important Remark: the vectors in V' are (linear) **functions**. The zero vector in V' is the **zero function**

$$0_{V'} : V \to F \ ; \ v \mapsto 0_V.$$

FACTS:

1) V' **is a vector space over** F, where we define addition and scalar multiplication as follows: let $T, S \in V'$. We want to define $T + S \in V'$ - thus, we need to define $T + S$ as a function (and make sure that it is linear!). We set

$$T + S : V \to F \ ; \ v \mapsto T(v) + S(v).$$

You should check that $T+S$ is indeed a linear map/function/transformation. For a scalar $c \in F$ and $T \in V'$ we define $cT \in V'$ to be

$$cT : V \to F \ ; \ v \mapsto c \cdot T(v),$$

where the '$\cdot$' denotes usual multiplication of scalars.

Remark: these may seem like 'obvious' definitions but we would technically need to check that all the axioms of a vector space are satisfied given these definitions of addition and scalar multiplication on V'.

2) $\dim V' = \dim V$, as we will see below.

Suppose for now that $V = F^n$. Then, in V there is an 'obvious' basis, namely the standard basis $S = (e_1, \dots, e_n)$. Let's give some examples of elements in the dual space: consider the following functions, for $i = 1, \dots, n$,

$$\phi_i : F^n \to F \ ; \ v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \mapsto a_i,$$

so that, in words, ϕ_i is the function that '*picks out the i^{th} entry of v*'. This is a linear function (map/transformation/whatever) as is easily verified - if we add two column vectors u and v and pick out the i^{th} entry, then that is the same as picking out the i^{th} entries of u and v and summing. A similar statement holds for scalar multiplication of vectors etc. Hence, $\phi_i \in V'$.

Let's show that the list $(\phi_1, \dots, \phi_n) \subset V'$ is linearly independent. Suppose that we have a linear relation

$$c_1\phi_i + \dots c_n\phi_n = 0_{V'}.$$

We want to show that $c_1 = \dots = c_n = 0$. The above equation is an equality of functions so that we must have

$$(c_1\phi_1 + \dots + c_n\phi_n)(v) = 0_{V'}(v) = 0 \in F, \quad \text{for every } v \in V.$$

This is the same as the infinite number of equations (one equation for each $v \in V$!) in the 'variables' $c_1, \dots, c_n$,

$$c_1\phi_1(v) + \dots + c_n\phi_n(n) = 0, \quad \text{for every } v \in V;$$

remember that $\phi_i(v)$ is a scalar! In particular, the above equation must hold for $v = e_1, e_2, \dots, e_n$, giving the equations

$$c_1\phi_1(e_i) + \dots + c_n\phi_i(e_n) = 0, \quad \text{for } i = 1, \dots, n.$$

Since $\phi_i(e_j)0$, if $i \neq j$, and $\phi_i(e_i) = 1$ (remember that ϕ_i picks out the i^{th} entry) we must have

$$c_1 = 0, \ c_2 = 0, \ \dots, \ c_n = 0.$$

Hence, the ϕ_i are linearly independent. This implies that $\dim V' \geq n$.

What about other examples of elements in V'? Choose some arbitrary vector $u \in V$. Then, we can consider the function

$$\beta_u : V \to F \ ; \ v \mapsto v \cdot u = v_1u_1 + \dots + v_nu_n,$$

where the output is the usual dot product of two vectors in F^n. Then, you should be able to check that $\beta_u \in V'$, for every $u \in V$ (ie, that β_u is linear). In fact,

every element of V' is of the form β_u, for a unique $u \in V$

We'll see this later on in the course.

Let's now show that $V' = \text{span}(\phi_1, \dots, \phi_n)$: thus, we are going to show that if $\alpha \in V'$ then there are scalars $c_1, \dots, c_n \in F$ such that

$$\alpha = c_1\phi_1 + \dots + c_n\phi_n,$$

an equality of functions. How do we find such c_i? Well, we'll cheat and I'll tell you what they are... set

$$c_1 = \alpha(e_1), \dots, c_n = \alpha(e_n) \in F.$$

Then, we are claiming that

$$\alpha = \alpha(e_1)\phi_1 + \dots + \alpha(e_n)\phi_n.$$

To check that this is true we use the following fact: (it might be a homework exercise...)

let $T, S \in L(V, W)$ be linear maps, and let $B = (v_1, \dots, v_n)$ be a basis. Then, $T = S$ if and only if

$$T(v_1) = S(v_1), \dots, T(v_n) = S(v_n).$$

We now apply this fact to the case $V = F^n, W = F$ to see that, in order to show that

$$\alpha = \alpha(e_1)\phi_1 + \dots + \alpha(e_n)\phi_n,$$

we need only show that

$$\alpha(e_i) = (\alpha(e_1)\phi_1 + \dots + \alpha(e_n)\phi_n)(e_i), \quad \text{for } i = 1, \dots, n.$$

But the RHS of this last equation is equal to

$$\alpha(e_1)\phi_1(e_i) + \dots + \alpha(e_n)\phi_n(e_i) = \alpha(e_i),$$

by definition of ϕ_i. Hence, $V' = \text{span}(\phi_1, \dots, \phi_n)$.

So, we have shown that $(\phi_1, \dots, \phi_n)$ is a basis of V', so that $\dim V' = n = \dim V$. This basis is called the **dual basis (of S)**.

We can generalise this to idea as follows: if $B = (v_1, \dots, v_n) \subset F^n$ is any basis of F^n then there exists a **dual basis of** B. It is defined as follows: since B is a basis any vector $v \in F^n$ can be written uniquely as

$$v = c_1 v_1 + \dots + c_n v_n,$$

for unique $c_1, \dots, c_n \in F$. Since the c's are unique we obtain a (linear) function

$$[-]_B : F^n \to F^n \; ; \; v \mapsto [v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

(This is the B-**coordinate map**). Notice that we have now defined a function (dependent upon B!)

$$\alpha_i^{(B)} : F^n \to F \; ; \; v \mapsto c_i$$

which is the function

'take a vector $v \in F^n$ to it's i^{th} coordinate (with respect to B)'

Hence, when we write $[v]_B$ we should really write

$$[v]_B = \begin{bmatrix} \alpha_1^{(B)}(v) \\ \vdots \\ \alpha_n^{(B)}(v) \end{bmatrix}$$

which, I must admit, looks pretty gross; but nevermind.

Now, the $\alpha_i^{(B)}$ are linear functions (check this! It follows because the B-coordinate map is linear) so that $\alpha_i^{(B)} \in (F^n)'$. Moreover, we have the following fact:

$$(\alpha_1^{(B)}, \dots, \alpha_n^{(B)}) \textbf{ is a basis of } (F^n)'$$

The proof is the same as the proof above: notice that $\alpha_i^{(S)} = \phi_i$, defined above. In particular, we have that any $\alpha \in (F^n)'$ can be written as

$$\alpha = \alpha(v_1)\alpha_1^{(B)} + \alpha(v_2)\alpha_2^{(B)} + \dots + \alpha(v_n)\alpha_n^{(B)}.$$

To check this you need only check that the LHS and RHS agree on the basis B.

Now, we can generalise the above considerations to any finite dimensional vector space V over F: let $B = (v_1, \dots, v_n) \subset V$ be a basis of the finite dimensional vector space V (over F). Then, the **dual basis of** B is the basis

$$(\alpha_1^{(B)}, \dots, \alpha_n^{(B)}) \subset V',$$

where $\alpha_i^{(B)}$ is the function that, given an input $v \in V$, picks out the i^{th} entry in the B-coordinate vector $[v]_B$. Moreover, if $\alpha \in V$ then we have

$$\alpha = \alpha(v_1)\alpha_1^{(B)} + ... + \alpha(v_n)\alpha_n^{(B)} \in V'.$$

As an example consider the basis

$$B = (v_1, v_2, v_3) = \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \subset \mathbb{C}^3.$$

Then, let's write $\phi_1 \in (\mathbb{C}^3)'$ as a linear combination of the dual basis of B, $(\alpha_1^{(B)}, \alpha_2^{(B)}, \alpha_3^{(B)})$: we have the formula

$$\phi_1 = \phi_1(v_1)\alpha_1^{(B)} + \phi_1(v_2)\alpha_2^{(B)} + \phi_1(v_3)\alpha_3^{(B)} = \alpha_1^{(B)} + \alpha_2^{(B)} + \alpha_3^{(B)}.$$