Worksheet 10/09. Math 110, Fall 2013.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Invariant Subspaces:

Throughout this worksheet V will always be a finite dimensional vector space over $F = \mathbb{R}, \mathbb{C}$.

1. Let $V = \mathbb{R}^2$ (over $F = \mathbb{R}$), $T \in L(V)$. Suppose that T admits an eigenvector v with eigenvalue $\lambda \in \mathbb{R}$. Let $u \notin \text{span}(v)$; explain why (u, v) is a basis of $\mathbb{R}^2$. Suppose that $T(u) = au + bv$. Prove that a is also an eigenvalue of T.

2. In Q1 suppose that $\lambda = a \neq 0$ so that T has only one distinct eigenvalue. Determine $w \in \mathbb{R}^2$ such that $T(w) = \lambda w + v$. In this case, what are the only T-invariant subspaces of $\mathbb{R}^2$? (Don't forget the obvious ones!)

3. Let (v, w) be as in Q2. Prove that (v, w) is a basis of $\mathbb{R}^2$ (ie, $w \notin \text{span}(v)$). What is the matrix of T with respect to (v, w)

4. Let $T \in L(V)$ and $U, W \subset V$ be T-invariant subspaces. Prove that $U + W$ is T-invariant.

5. Let $T \in L(\mathbb{R}^2)$. Suppose that T admits two distinct proper T-invariant subspaces of $\mathbb{R}^2$ ($U \subset V$ is a *proper subspace of V* if U is a subspace and $U \neq \{0\}, V$). Prove that there is a basis of $\mathbb{R}^2$ such that the matrix of T with respect to this basis is diagonal.

6. Let $T \in L(\mathbb{R}^3)$ be such that there is some $0 \neq v \in \mathbb{R}^3$ with $T^3(v) = \underline{0}$, while $T^2(v) \neq \underline{0}$.

- show that $T(v) \neq \underline{0}$.
- show that the only eigenvalue of T is $\lambda = 0$.
- show that $B = (v, T(v), T^2(v))$ is linearly independent, hence a basis of $\mathbb{R}^3$.
- what is the matrix of T with respect to B?

Consider the linear map

$$T : \mathbb{R}^3 \to \mathbb{R}^3 \; ; \; \underline{v} \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \underline{v}$$

Prove that there is a basis of $\mathbb{R}^3$ such that the matrix of T with respect this basis is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$