Worksheet 3/31. Math 110, Spring 2014. SOLUTIONS

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Remember that $F \in \{\mathbb{R}, \mathbb{C}\}$. Send me an email if you have any questions!

Eigenstuff

1. Consider the operator

$$T : \mathbb{C}^2 \to \mathbb{C}^2 \; ; \; \underline{x} \mapsto \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \underline{x}$$

i) Find a polynomial relation of the form

$$aT^2 + bT + c\mathrm{id}_{F^2} = 0,$$

where $\mathrm{id}_{\mathbb{C}^2}$ is the identity operator on $\mathbb{C}^2$ and 0 is the zero operator. This means that the LHS of the above equation should be equal to the zero operator so that, for any $\underline{x} \in \mathbb{C}^2$, we have

$$(aT^2 + bT + c\mathrm{id}_{\mathbb{C}^2})(\underline{x}) = 0_{\mathbb{C}^2} \in \mathbb{C}^2.$$

ii) Suppose that $c \in \mathbb{C}$ is an eigenvalue of T (assuming it exists!). Using i) determine the possible values of c. (Hint: the eigenvalues are $c = 0, 2$)

iii) Show that $\mathrm{null}(T) \neq \{0\}$. Find an eigenvector v of T with associated eigenvalue 0.

iv) Let $u \notin \mathrm{null}(T)$. Explain why $T(u)$ is an eigenvector of T with associated eigenvalue 2. (Hint: use i) above) Explain why $(v, T(u))$ is a basis of $\mathbb{C}^2$. What is the matrix of T with respect to this basis?

v) Let $T \in L(\mathbb{C}^2)$. Suppose that T satisfies the polynomial relation

$$aT^2 + bT + c\mathrm{id}_{\mathbb{C}^2} = 0.$$

By considering what you have shown in i)-iv) above, determine a method that allows you to find an eigenvector of T <u>without</u> performing any calculations. Does this method work if $F = \mathbb{R}$? (Answer: No) Describe a condition on a, b, c so that this method will work if $F = \mathbb{R}$.

Solution:

i) We have $T^2 - 2T = 0$.

ii) If c is an eigenvalue and v is an eigenvector then

$$0 = (T^2 - 2T)(v) = (c^2 - 2c)v \implies c^2 = 2c \implies c = 0, 2.$$

iii) Let $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Then, $v \in \mathrm{null}(T)$.

iv) We have $0 = (T-2)(T(u))$ so that $T(T(u)) = 2T(u)$, and since $T(u) \neq 0$ then $T(u)$ is an eigenvector with eigenvalue 2. Since v and $T(u)$ are eigenvectors corresponding to distinct eigenvalues then $(v, T(u))$ is linearly independent, hence a basis. The matrix of T with respect to $B = (v, T(u))$ is

$$[T]_B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

2. Recall that an operator $T \in L(V)$ is diagonalisable if there exists a basis B of V consisting of eigenvectors of T. Assume that $F = \mathbb{C}$. Give examples of the following types of operators or explain why such an operator can't exist: (look for $T \in L(F^2)$, defined by a 2×2 matrix A)

i) diagonalisable, invertible,

ii) diagonalisable, not invertible,

iii) not diagonalisable, invertible,

iv) not diagonalisable, not invertible.

Do you think there is a relationship between invertibility and diagonalisability?

Solution: We have $T : F^2 \to F^2$; $x \mapsto Ax$, where

i) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$,

ii) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$,

iii) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$,

iv) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

3. An operator $T \in L(V)$ is called **nilpotent** if $T^n = 0$, for some n. Suppose that $T \in L(V)$ is nilpotent and such that $T^n = 0$, while $T^{n-1} \neq 0$, where $n = \dim V$. Find a vector $v \in V$ (there are infinitely many!) such that $(v, T(v), T^2(v), \dots, T^{n-1}(v)) \subset V$ is linearly independent. Determine the matrix of T with respect to this basis. (Perhaps try $\dim V = 2, 3$ to see the pattern) What are the eigenvalues of nilpotent T?

Solution: Take $v \in V$ such that $v \notin \operatorname{null} T^{n-1}$ - such a v exists since $T^{n-1} \neq 0$. Then, suppose that

$$c_1 v + \dots + c_n T^{n-1}(v) = 0.$$

Apply T^{n-1} to the above equation to obtain $c_1 T^{n-1}(v) = 0$, so that $c_1 = 0$. Apply T^{n-2} to the equation

$$c_2 T(v) + \dots + c_n T^{n-1}(v) = 0$$

to show that $c_2 = 0$. Continue in this manner to obtain $c_1 = \dots = c_n = 0$. The matrix is

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

4. Let $\mathfrak{sl}_2 = \{T \in L(\mathbb{C}^2) \mid trT = 0\}$, where $\mathrm{tr}\, T = \mathrm{tr} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$, where the matrix appearing is the matrix of T with respect to the standard basis. In this problem you will need the following fact:

$$\text{if } T \in L(\mathbb{C}^2) \text{ then } T^2 - (\mathrm{tr}\, T) \cdot T + (\det T) \cdot \mathrm{id}_{\mathbb{C}^2} = 0.$$

i) Prove: if $T \in \mathfrak{sl}_2$ then T is either nilpotent or diagonalisable. (Hint: you'll need to use the fact above!) Show that there is exactly one operator in $\mathfrak{sl}_2$ that is both nilpotent and diagonalisable.

ii) Find a nilpotent, non-diagonalisable operator $T \in \mathfrak{sl}_2$.

Solution:

i) We have $T^2 + (\det T)\mathrm{id}_{\mathbb{C}^2} = 0 \in L(V)$. If $\det T = 0$ then T is nilpotent. Else, we have

$$(T - \sqrt{\det T}\mathrm{id}_{\mathbb{C}^2})(T + \sqrt{\det T}\mathrm{id}_{C^2}) = 0,$$

so that either $T = \pm\sqrt{\det T}\mathrm{id}_{\mathbb{C}^2}$, or $T \neq \pm\sqrt{\det T}\mathrm{id}_{\mathbb{C}^2}$ and we can use the method of Problem 1 above to show that T has two distinct eigenvalues, hence is diagonalisable. The only nilpotent operator that is also diagonalisable is $T = 0$.

ii) For example, we can take

$$T : \mathbb{C}^2 \to \mathbb{C}^2 \; ; \; x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$

5. Consider the following operators

$$S_1 : \mathbb{C}^4 \to \mathbb{C}^4 \; ; \; \underline{x} \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underline{x}, \quad S_3 : \mathbb{C}^4 \to \mathbb{C}^4 \; ; \; \underline{x} \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underline{x}$$

i) Verify that $S^2 = \mathrm{id}_{\mathbb{C}^2}$ and $R^2 = \mathrm{id}_{\mathbb{C}^2}$ and that $S_1S_3 = S_3S_1$.

ii) What are the allowed eigenvalues of S and R? Show that both S and R must admit two distinct eigenvalues $1, -1$.

iii) Suppose that $S_1(w) = -w$. Show that $S_3(w)$ is also an eigenvector of S_1 with eigenvalue -1. Show that $\dim \mathrm{null}(S_1 + \mathrm{id}_{\mathbb{C}^2}) = 1$. Show that $S_3(w) = w$.

iv) Determine a basis B of $\mathrm{null}(S_1 - \mathrm{id}_{\mathbb{C}^4})$ (Hint: $\dim \mathrm{null}(S_1 - \mathrm{id}_{\mathbb{C}^3}) = 3$)

v) Explain why $\mathrm{null}(S_1 - \mathrm{id}_{\mathbb{C}^4})$ is S_3-invariant.

vi) Consider the restriction operator

$$S_3' : \mathrm{null}(S_1 - \mathrm{id}_{\mathbb{C}^4}) \to \mathrm{null}(S_1 - \mathrm{id}_{\mathbb{C}^4}) \; ; \; \underline{x} \mapsto S_3(\underline{x}).$$

Find the matrix $[S_3']_B$ of S_3' with respect to B. (Hint: it should be 3×3)

vii) Find two distinct eigenvalues and three linearly independent eigenvectors (v_1, v_2, v_3) of S_3'.

viii) Show that $C = (w, v_1, v_2, v_3)$ is linearly independent, and determine $[S_1]_C, [S_3]_C$. What do you notice?

Solution: Call the two matrices appearing above A and B.

i) Easy.

ii) Allowed eigenvalues are ± 1 - ie roots of $\lambda^2 - 1 = 0$. It's obvious that e_3 is an eigenvector of S_1 with eigenvalue 1, and e_1 is an eigenvector of S_3 with eigenvalue 1. Now, since e_1 is not an eigenvector of S_1 then $(S_1 - 1)(e_1) = -e_1 + e_2$ is an eigenvector of S_1 with eigenvalue -1 (Problem 1 above). Similarly, we see that $-e_3 + e_4$ is an eigenvector of S_3. Hence, S_1 and S_3 have two distinct eigenvalues.

iii) $S_1(S_3(w)) = S_3(S_1(w)) = -S_3(w)$. We consider the matrix

$$A + I_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, we have

$$\mathrm{null}(S_1 + \mathrm{id}_{\mathbb{C}^4}) = \mathrm{null}(A + I_4),$$

and there is one free variable in the reduced echelon form above so that

$$1 = \dim \mathrm{null}(A + I_4) = \dim \mathrm{null}(S_1 + \mathrm{id}_{\mathbb{C}^4}).$$

In fact, we have $\mathrm{null}(S_1 + \mathrm{id}_{\mathbb{C}^4}) = \mathrm{span}(-e_1 + e_2)$. It's easy to see that $S_3(w) = w$ if $w \in \mathrm{span}(-e_1 + e_2)$.

iv) We have

$$\mathrm{null}(S_1 - \mathrm{id}_{\mathbb{C}^4}) = \mathrm{span}(e_1 + e_2, e_3, e_4).$$

v) Use that $S_3 S_1 = S_1 S_3$.

vi) USe the basis B described in iv) to see that

$$[S_3']_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

vii) We have eigenvalues ± 1 and eigenvectors

$$(e_1 + e_2, e_3 + e_4, -e_3 + e_4).$$

viii) We have

$$[S_1]_C = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, [S_3]_C = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}.$$

They are simultaneously diagonal.