Worksheet 3/5. Math 110, Spring 2014.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Remember that $F \in \{\mathbb{R}, \mathbb{C}\}$. Send me an email if you have any questions!

Matrices and linear maps

1. Determine the matrices $[T]_B^C$ of the linear map $T : V \to W$ respect to the bases $B \subset V, C \subset W$.

i) $V = W = F^3$, $B = (e_1, e_2, e_3)$, $C = \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$

$$T : V \to W \; ; \; \underline{x} \mapsto \begin{bmatrix} x_1 - 2x_3 \\ x_1 + x_2 + x_3 \\ x_2 - x_3 \end{bmatrix}$$

ii) $V = W = F^3$, $B = \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$, $C = (e_1, e_2, e_3)$

$$T : V \to W \; ; \; \underline{x} \mapsto \begin{bmatrix} x_1 - 2x_3 \\ x_1 + x_2 + x_3 \\ x_2 - x_3 \end{bmatrix}$$

iii) $V = F^3, W = F^2$, $B = \left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$, $C = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$

$$T : V \to W \; ; \; \underline{x} \mapsto A\underline{x}, \text{ where } A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

iv) $V = P_2(F), W = F^3$, $B = (1, x - 1, x^2 + 1)$, $C = \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$

$$T : V \to W \; ; \; p \mapsto \begin{bmatrix} p(1) \\ p(-1) + p(0) \\ -p(2) \end{bmatrix}$$

Solution:

i) Let $C = (v_1, v_2, v_3)$. We need to write $T(e_i)$ as a linear combination for the basis vectors in C, for $i =, 1, 2, 3$. Then, the corresponding weights will give the columns of the matrix we are looking for. So, we see that

$$T(e_1) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \text{we need scalars } a_1, a_2, a_3 \text{ such that } a_1 v_1 + a_2 v_2 + a_3 v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Hence, we must solve the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \underline{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

This is Math 54... We find, by row-reduction methods, that we must have $a_1 = -1, a_2 = 2, a_3 = 0$. Similarly, for $T(e_2) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, we must have $a_1 = -1, a_2 = 2, a_3 = -1$; and for $T(e_3) = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$ we have $a_1 = -1, a_2 = -2, a_3 = 1$. Hence, the desired matrix is

$$[T]_B^C = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

ii) We proceed in exactly the same way as above: we find that

$$[T]_B^C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

iii) Solution forthcoming (you do not need it for the Midterm)

iv) Solution forthcoming (you do not need it for the Midterm)

2. Suppose that $B \subset V, C \subset W$ are bases and $T : V \to W$ is a linear map.

   i) Suppose that $[T]_B^C$ has exactly two rows consisting of zeroes. What can you deduce about $\dim \text{range}(T)$? $\dim \text{null}(T)$?

   ii) Suppose that $[T]_B^C$ has exactly four columns consisting of zeroes. What can you deduce about $\dim \text{range}(T)$? $\dim \text{null}(T)$?

   iii) Suppose that $\dim V = \dim W = 10$ and that $[T]_B^C$ admits six rows consisting of zeroes and five columns consisting of zeroes. Is it possible that the remaining nonzero columns are linearly independent?

Solution:

i) Since there are two rows of zeroes we must have that two of the basis vectors in C do not lie in range(T) - if the rows are row i and j then the basis vectors $w_i, w_j \notin \text{range}(T)$. Hence, we must have $\dim \text{range}(T) \leq \dim W - 2$. This implies that $\dim \text{null}(T) = \dim V - \dim \text{range}(T) \geq \dim V - \dim W + 2$.

ii) If there are four columns consisting of zeroes then four of the basis vectors in B are mapped to the zero vector in W - hence, we must have that $\dim \text{null}(T) \geq 4$, so that $\dim \text{range}(T) = \dim V - \dim \text{null}(T) \leq \dim V - 4$.

iii) If we reorder the elements in C then we can assume that the zero rows are the alst six. Hence, $[T]_B^C$ has nonzero entries only appearing in the first four rows. Moreover, five of the columns are zero so that we have five $(= 10 - 5)$ nonzero columns, with entries only appearing in the first four rows, ie, we have five vectors in $\text{span}(e_1, \dots, e_4)$. Hence, they must be linearly dependent.

3.

i) Verify that $C = \left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$ is a basis of F^4.

ii) What is the matrix of
$$\text{id}_{F^4} : F^4 \to F^4 \ ; \ v \mapsto v,$$
with respect to $B = (e_1, e_2, e_3, e_4)$ and C?

iii) Consider the linear map
$$T : F^4 \to F^4 \ ; \ \underline{x} \mapsto A\underline{x}, \text{ where } A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & -2 & 3 & 1 \end{bmatrix}.$$
What is the matrix $[T]_B^B$?

iv) Recall that, if $T : V \to W, S : W \to X$ are linear maps and $B \subset V, C \subset W, D \subset X$ are bases then
$$[S \circ T]_B^D = [S]_C^D [T]_B^C.$$
What is $[T]_B^C$, where T, B and C are as in *i*), *ii*), *iii*).

v) If P is an invertible 4×4 matrix, we can consider
$$P = [\text{id}_{F^4}]_B^{C'},$$
for some basis C'; can you say which basis? (Your answer will involve the word 'columns'... You may also have to recall some Math 54 material)

What is $[T]_B^{C'}$?

Solution:

i) Row-reduce the matrix with the given vectors as its columns and verify that there is a pivot in every column/row.

ii) It is the matrix whose columns are the vectors in C appearing in the order they appear in C.

iii) It is the matrix A.

iv)

v) C' consists of the columns of the inverse of P. We have
$$[T]_B^{C'} = [\text{id} \circ T]_B^{C'} = [\text{id}]_B^{C'} [T]_B^B = PA.$$