Worksheet 3/5. Math 110, Spring 2014.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Remember that $F \in \{\mathbb{R}, \mathbb{C}\}$. Send me an email if you have any questions!

Matrices and linear maps

1. Determine the matrices $[T]_B^C$ of the linear map $T : V \to W$ respect to the bases $B \subset V, C \subset W$.

i) $V = W = F^3$, $B = (e_1, e_2, e_3)$, $C = \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$

$$T : V \to W \; ; \; \underline{x} \mapsto \begin{bmatrix} x_1 - 2x_3 \\ x_1 + x_2 + x_3 \\ x_2 - x_3 \end{bmatrix}$$

ii) $V = W = F^3$, $B = \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$, $C = (e_1, e_2, e_3)$

$$T : V \to W \; ; \; \underline{x} \mapsto \begin{bmatrix} x_1 - 2x_3 \\ x_1 + x_2 + x_3 \\ x_2 - x_3 \end{bmatrix}$$

iii) $V = F^3, W = F^2$, $B = \left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$, $C = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$

$$T : V \to W \; ; \; \underline{x} \mapsto A\underline{x}, \text{ where } A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

iv) $V = P_2(F), W = F^3$, $B = (1, x - 1, x^2 + 1)$, $C = \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$

$$T : V \to W \; ; \; p \mapsto \begin{bmatrix} p(1) \\ p(-1) + p(0) \\ -p(2) \end{bmatrix}$$

2. Suppose that $B \subset V, C \subset W$ are bases and $T : V \to W$ is a linear map.

i) Suppose that $[T]_B^C$ has exactly two rows consisting of zeroes. What can you deduce about $\dim \operatorname{range}(T)$? $\dim \operatorname{null}(T)$?

ii) Suppose that $[T]_B^C$ has exactly four columns consisting of zeroes. What can you deduce about $\dim \operatorname{range}(T)$? $\dim \operatorname{null}(T)$?

iii) Suppose that $\dim V = \dim W = 10$ and that $[T]_B^C$ admits six rows consisting of zeroes and five columns consisting of zeroes. Is it possible that the remaining nonzero columns are linearly independent?

3.

i) Verify that $C = \left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$ is a basis of F^4.

ii) What is the matrix of

$$\mathrm{id}_{F^4} : F^4 \to F^4 \ ; \ v \mapsto v,$$

with respect to $B = (e_1, e_2, e_3, e_4)$ and C?

iii) Consider the linear map

$$T : F^4 \to F^4 \ ; \ \underline{x} \mapsto A\underline{x}, \ \text{where } A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & -2 & 3 & 1 \end{bmatrix}.$$

What is the matrix $[T]_B^B$?

iv) Recall that, if $T : V \to W, S : W \to X$ are linear maps and $B \subset V, C \subset W, D \subset X$ are bases then

$$[S \circ T]_B^D = [S]_C^D [T]_B^C.$$

What is $[T]_B^C$, where T, B and C are as in *i*), *ii*), *iii*).

v) If P is an invertible 4×4 matrix, we can consider

$$P = [\mathrm{id}_{F^4}]_B^{C'},$$

for some basis C'; can you say which basis? (Your answer will involve the word 'columns'... You may also have to recall some Math 54 material)

What is $[T]_B^{C'}$?