Worksheet 2/26. Math 110, Spring 2014. Solutions

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Remember that $F \in \{\mathbb{R}, \mathbb{C}\}$. Send me an email if you have any questions!

Dual things; annhilators

1. Consider the linear map

$$T : F^3 \to F^4 \; ; \; v = \begin{bmatrix} x_1(v) \\ x_2(v) \\ x_3(v) \end{bmatrix} \mapsto \begin{bmatrix} x_1(v) - 2x_3(v) \\ 0 \\ x_1(v) + x_2(v) + x_3(v) \\ 3x_2(v) + x_3(v) \end{bmatrix}.$$

i) Let $S_4 \subset F^4$ be the standard basis, and let $S_4' = (y_1, \dots, y_4)$ be the dual basis of S_4'; similarly, let $S_3 \subset F^3$ be the standard basis and let $S_3' = (x_1, x_2, x_3)$ be the dual basis of S_3. What is $T'(y_i)$, for each $i = 1, \dots, 4$? (Here $T' \in L((F^4)', (F^3)')$ is the dual map of T)

ii) Consider the linear functional

$$\alpha : F^4 \to F \; ; \; v = \begin{bmatrix} y_1(v) \\ y_2(v) \\ y_3(v) \\ y_4(v) \end{bmatrix} \mapsto -y_1(v) + 3y_3(v) + y_4(v).$$

Write α as a linear combination of S_4', ie, find scalars $a, b, c, d \in F$ such that

$$\alpha = ay_1 + by_2 + cy_3 + dy_4.$$

iii) Write $T'(\alpha) \in (F^3)'$ as a linear combination of S_3', ie, find scalars $p, q, r \in F$ such that

$$T'(\alpha) = px_1 + qx_2 + rx_3.$$

Solution:

i) Recall that we have $T'(y_i)$ is the linear functional on F^3 defined as the composition

$$F^3 \xrightarrow{T} F^4 \xrightarrow{y_i} F$$

So, in order to understand this linear map it suffices to write it in terms of the dual basis S_3': thus we have

$$T'(y_i) = a_1x_1 + a_2x_2 + a_3x_3, \; a_1, a_2, a_3 \in F$$

Recall that we can determine the scalars a_j by evaluating $T'(y_i)(e_j)$, where $e_j \in S_3$ are the standard basis vectors of F^3. Thus, we have (for $i = 1$)

$$T'(y_1)(e_1) = (y_1 \circ T)(e_1) = 1, \; T'(y_1)(e_2) = (y_1 \circ T)(e_2) = 0,$$

$$T'(y_1)(e_3) = (y_1 \circ T)(e_1) = -2.$$

Hence, we have

$$T'(y_1) = x_1 - 2x_3.$$

Similarly, we have

$$T'(y_2)(e_1) = (y_1 \circ T)(e_1) = 0, \; T'(y_2)(e_2) = (y_2 \circ T)(e_2) = 0,$$
$$T'(y_2)(e_3) = (y_2 \circ T)(e_1) = 0.$$

Hence,

$$T'(y_2) = 0.$$

We also find

$$T'(y_3) = x_1 + x_2 + x_3, \; T'(y_4) = 3x_2 + x_3.$$

ii) Remember that we have

$$\alpha(e_1) = a, \; \alpha(e_2) = b, \; \alpha(e_3) = c, \; \alpha(e_4) = d.$$

Hence, we have

$$\alpha = -y_1 + 3y_3 + y_4.$$

iii) The dual linear map is, funnily enough, linear! Hence, we have

$$T'(\alpha) = T'(-y_1 + 3y_3 + y_4) = -T'(y_1) + 3T'(y_3) + T'(y_4).$$

Using i) we find

$$T'(\alpha) = 2x_1 + 6x_2 + 6x_3.$$

2. Do the same problems above (with appropriate adjustments!) for the linear map

$$T : F^3 \to F^2 \; ; \; v = \begin{bmatrix} x_1(v) \\ x_2(v) \\ x_3(v) \end{bmatrix} \mapsto \begin{bmatrix} x_1(v) + x_2(v) - x_3(v) \\ x_1(v) + 2x_2(v) + 3x_3(v) \end{bmatrix},$$

and the linear functional

$$\alpha : F^2 \to F \; ; \; v = \begin{bmatrix} y_1(v) \\ y_2(v) \end{bmatrix} \mapsto y_1(v).$$

Solution: You have to follow the same procedure as Q1 - ie, do parts i), ii), iii). You should find that

$$T'(\alpha) = x_1 + x_2 - x_3.$$

3. **FACT:** Elements of $(F^n)'$ may be thought of as the vector space $Mat_{1,n}(F)$ of $1 \times n$ matrices (with entries in F). Make this identification.

i) Given a vector $v \in F^n$ and $A \in Mat_{1,n}(F)$, verify that Av can be considered as a scalar.

ii) Let $B = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right) \subset \mathbb{C}^2$, a basis of $\mathbb{C}^2$. What are the row-vectors (ie, $1 \times n$ matrices) $A_1, A_2 \in Mat_{1,n}(\mathbb{C})$ such that (A_1, A_2) is the dual basis of B? (*Hint: what equations must A_1 and A_2 satisfy?*)

iii) Write A_1, A_2 as linear combinations of elements of the dual basis of S_2 (the standard basis of $\mathbb{C}^2$). (You'll need to think which row vectors the elements of S_2' correspond to...)

iv) Consider the linear map

$$T : \mathbb{C}^2 \to \mathbb{C}^2 \ ; \ v \mapsto \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} v.$$

If we denote $S_2' = (x_1, x_2)$, then write $T'(x_1)$ and $T'(x_2)$ as linear combinations of S_2'. Can you find a matrix $A \in Mat_2(\mathbb{C})$ such that

$$T' : (\mathbb{C}^2)' \to (\mathbb{C}^2)' \ ; \ \alpha \mapsto \alpha A?$$

Here we are considering elements of $(\mathbb{C}^2)'$ as row-vectors. Is the matrix A related to the matrix defining T in any way?

Solution:

i) A is a $1 \times n$ matrix, and v is a $n \times 1$ matrix. Hence, their product Av is a 1×1 matrix, which is the same things as a scalar.

ii) Denote $B = (v_1, v_2)$ Then, we must have that

$$A_1 v_1 = 1, \ A_1 v_2 = 0, \ A_2 v_1 = 0, \ A_2 v_2 = 1.$$

So, if we denote $A_1 = [a \ b]$ and $A_2 = [c \ d]$ then we need

$$a + b = 1, \ -a - 3b = 0, \ c + d = 0, \ -c - 3d = 1.$$

We can write these equations in the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, we need to determine the inverse of the matrix

$$\begin{bmatrix} 1 & -1 \\ 1 & -3 \end{bmatrix};$$

this is the matrix

$$-\frac{1}{2} \begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix},$$

so that

$$A_1 = \begin{bmatrix} \frac{3}{2} & \frac{-1}{2} \end{bmatrix}, \ A_2 = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$$

iii) Since we have

$$A_1 = A_1(e_1)x_1 + A_1(e_2)x_2, \ A_2 = A_2(e_1)x_1 + A_2(e_2)x_2,$$

we see that

$$A_1 = \frac{3}{2}x_1 - \frac{1}{2}x_2, \ A_2 = \frac{1}{2}x_1 - \frac{1}{2}x_2.$$

So, we notice that the row vector [1 0] corresponds to the dual basis vector x_1, and [0 1] corresponds to x_2.

iv) PRoceeding as in Q1, Q2, you will find that

$$T'(x_1) = x_1 + 2x_2, \;\; T'(x_2) = -x_2.$$

Thus, if we let

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix},$$

then we have that the row vector αA corresponds to $T'(\alpha)$. Thus, we have that this matrix A is the transpose of the matrix defining T.

4. Let

$$U = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in F^4 \;\middle|\; \begin{matrix} v_1 + v_4 = 0 \\ v_2 - v_3 = 0 \end{matrix} \right\} \subset F^4;$$

this is a subspace of F^4.

i) Determine a basis $B = (u_1, u_2)$ of U. Extend this basis to a basis C of F^4.

ii) Let $C' = (\alpha_1, \dots, \alpha_4)$ be the dual basis of C. Write α_i as a linear combination of the elements of $S_4' = (x_1, x_2, x_3, x_4)$ (the dual basis of the standard basis of F^4).

iii) Prove that (α_3, α_4) is a basis of U°, the annhilator of U. (*Hint: you don't need to show that these functionals span U°!*)

iv) Prove that $x_1 + x_4, x_2 - x_3 \in U^\circ$.

v) (*Harder*) Suppose that A is an $m \times n$ matrix with entries in F. Suppose that $U = \text{null}(A) \subset F^n$. Can you find a spanning list of U°? Can you find a basis of U°?

Solution:

i) We see that if $\underline{v} \in U$ then we have

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -v_4 \\ v_3 \\ v_3 \\ v_4 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + v_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the two vectors above must span U. Moreover, they are linearly independent (obviously) so that they form a basis (u_1, u_2), where

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \;\; u_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

You can check that the list (u_1, u_2, e_1, e_2) is linearly independent (by row-reducing the matrix whose columns are these vectors and verifying there is a pivot in each row, for example).

ii) We have

$$\alpha_1 = \alpha_1(e_1)x_1 + \alpha_1(e_2)x_2 + \alpha_1(e_3)x_3 + \alpha_1(e_4)x_4$$

Since we have

$$1 = \alpha_1(u_1) = \alpha_1(e_2 + e_3) = \alpha_1(e_2) + \alpha_1(e_3),$$

and $\alpha_1(e_2) = 0$ - because e_2 is a vector in the basis C, and using the definition of the dual basis C' - we have $\alpha_1(e_3) = 1$. Similarly, we find that $\alpha_1(e_4) = 0$ - using that $\alpha_1(u_2) = 0$. Hence, $\alpha_1 = x_3$.

In a similar way we find $\alpha_2 = x_4$.

Now, we have that

$$\alpha_3 = \alpha_3(e_1)x_1 + \alpha_3(e_2)x_2 + \alpha_3(e_3)x_3 + \alpha_3(e_4)x_4$$

So, we must have

$$0 = \alpha_3(u_1) = \alpha_3(e_2 + e_3) = \alpha_3(e_2) + \alpha_3(e_3) = 0 + \alpha_3(e_3),$$

because $\alpha_3(e_2) = 0$ - α_3 is the dual basis vector of e_1, when we consider e_1 as a vector in the basis C. Similarly, we have

$$0 = \alpha_3(u_2) = \alpha_3(-e_1 + e_4) = -\alpha_3(e_1) + \alpha_3(e_4) = -1 + \alpha_3(e_4) \implies \alpha_3(e_4) = 1.$$

Hence, $\alpha_3 = x_1 + x_4$. Proceeding in a similar manner we find $\alpha_4 = x_2 - x_3$.

iii) You can verify that

$$\alpha_3(u_1) = \alpha_3(u_2) = \alpha_4(u_1) = \alpha_4(u_2) = 0.$$

Hence, we have $\alpha_3, \alpha_4 \in U^\circ$ and, since (α_3, α_4) is linearly independent and $\dim U^\circ = 4 - \dim U = 2$, we have that (α_3, α_4) is a basis of U°.

iv) This is trivial, by part iii).

v) Note that we have

$$U = \text{null}\left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}\right)$$

and that we've seen that $x_1 + x_4, x_2 - x_3$ are a basis of U°. These functionals correspond to the row vectors $[1\ 0\ 0\ 1]$ and $[0\ 1\ -1\ 0]$ respectively. Hence, we guess(!) that in the general case, if we have $A = [a_{ij}]$ then

$$\alpha_1 = \sum_{j=1}^{n} a_{1j}x_j,\ \dots\ , \alpha_m = \sum_{j=1}^{n} a_{mj}x_j$$

are elements of U° - ie, the rows of A correspond to a spanning list of U°. This is indeed true (but I will not prove it - give it a go yourself). To determine a basis of U° we row reduce A to reduced echelon form, then the rows of the reduced echelon form will correspond to a basis of U° - can you see why? It's enough to note that the rows will be linearly independent and that $\dim U^\circ = \dim \text{col}(A)$ (why?).