Worksheet 2/5. Math 110, Spring 2014. Solutions

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Linear independence, span, bases

1. Determine which of the following lists $(v_1, \dots, v_m)$ in $V = F^n$ are

a) linearly independent, b) spanning lists (of V).

Also, say whether the lists can be a basis of V (with appropriate justification).
If the list is not linear independent find a nontrivial linear relation among $(v_1, \dots, v_m)$.

i) $V = \mathbb{R}^3$,

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

ii) $V = \mathbb{C}^3$

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

(Do you need to do anything new here?)

iii) $V = \mathbb{C}^4$

$$v_1 = \begin{bmatrix} \sqrt{-1} \\ 0 \\ 1 \\ -1 \end{bmatrix}, \; v_2 = \begin{bmatrix} -2 \\ 1 \\ \sqrt{-1} \\ 1+\sqrt{-1} \end{bmatrix}, \; v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

iv) $V = \mathbb{C}^2$ (considered as a vector space over $\mathbb{R}$... only allowed $\mathbb{R}$ scalars)

$$v_1 = \begin{bmatrix} \sqrt{-1} \\ -\sqrt{-2} \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}, \; v_3 = \begin{bmatrix} 0 \\ -4 \end{bmatrix}.$$

Solution: In order to determine whether the following lists are spanning lists and/or linearly independent we form the matrix $[v_1 \cdots v_k]$ and row-reduce: if there is a pivot in each column then the list is linearly independent; if there is a pivot in each row then the list is spanning. Well, this is not *correct* as we'll see...

i) We have

$$[v_1 \; v_2 \; v_3] = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, as there is a pivot in each row and column the list is linearly independent and spanning (hence a basis of $\mathbb{R}^3$).

ii) This has the same solution as above - when we perform our row-reductions we are only using rational numbers so that our row-reductions do not depend on whether we are considering the columns of the matrix in $\mathbb{R}^3$ or $\mathbb{C}^3$. If we had to use complex scalars in our row-reduction then we would not be able to row-reduce the above matrix over $\mathbb{R}$.

iii) First we observe it is impossible for there to exist a pivot in each row (each pivot necessarily must lie in a column of the below matrix so that there are at most three pivots; we'd need four pivots for the list to be spanning). We have

$$[v_1\ v_2\ v_3] = \begin{bmatrix} \sqrt{-1} & -2 & -3 \\ 0 & 1 & 1 \\ 1 & \sqrt{-1} & 0 \\ -1 & 1+\sqrt{-1} & 1 \end{bmatrix} \sim \begin{bmatrix} \sqrt{-1} & -2 & -3 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & \sqrt{-1}-3 & \sqrt{-1}-3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the list is linearly independent.

iv) We row-reduce the following matrix (using only $\mathbb{R}$-scalars, ie, we can only scale a row by an element of $\mathbb{R}$):

$$\begin{bmatrix} \sqrt{-1} & 1 & 0 \\ -\sqrt{2}\sqrt{-1} & -\sqrt{2} & -4 \end{bmatrix} \sim \begin{bmatrix} \sqrt{-1} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that we now have a 'pivot' whose value is a complex number! AAHHHH!! Don't worry, this is allowed, it just means that we now have to solve the linear equations (attempting to find $\mathbb{R}$ solutions)

$$\begin{aligned} \sqrt{-1}x_1 + x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

The only solution is $x_1 = x_2 = x_3 = 0$: there are no nonzero real solutions to the equation

$$\sqrt{-1}x_1 + x_2 = 0,$$

otherwise there would exist a real number (namely $-x_2/x_1$) equal to $\sqrt{-1}$! Hence, the given list is linearly independent. However, it is not a spanning list since $\mathbb{C}^2$ is four dimensional (over $\mathbb{R}$) and we only have three vectors in our list.

2. For the last example in Q1, why are you not allowed to use the following 'fact'?

if $m > \dim V$ then $(v_1, \ldots, v_m)$ is linearly dependent.

Solution: Because we are considering $\mathbb{C}^2$ as a real vector space - its dimension (as a real vector space) is 4. Thus, we are not allowed to use the above 'fact'.

3. Suppose that $(v_1, \ldots, v_m) \subset V$ is linearly independent. Prove that $(v_1, \ldots, v_k)$ is linearly independent for any $1 \leq k \leq m$. Can you generalise this to show that any sublist $(v_{i_1}, \ldots, v_{i_k})$ is linearly independent? (Hint: yes, you can! But can *you* do it?)

Is it true that if $(v_1, \dots, v_m) \subset V$ is linearly dependent then any sublist is linearly dependent? (*Hint: think of some examples of (nontrivial) linearly dependent sets you know.*)

Solution: Suppose that we have a linear relation

$$c_1 v_1 + \dots + c_k v_k = 0_V.$$

Then, we can rewrite this relation as

$$c_1 v_1 + \dots + c_k v_k + 0 v_{k+1} + \dots + 0 v_m = 0_V.$$

Hence, since $(v_1, \dots, v_m)$ is linearly independent we must have all coefficient are zero. In particular, we must have $c_1 = c_2 = \dots = c_k = 0$. Hence, $(v_1, \dots, v_k)$ is linearly independent.

To generalise, we can say the following: let $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$. Then, the list $(v_{i_1}, \dots, v_{i_k})$ is linearly independent. In words,

<u>any</u> sublist of a linearly independent list is linearly independent

4. Prove or give a counterexample: if $(v_1, \dots, v_m) \subset V$ is linearly independent and (u, v) is linearly indpendent, with $u, v \notin \text{span}(v_1, \dots, v_m)$, then $(v_1, \dots, v_m, u, v)$ is linearly independent.

Solution: This is false: consider the linearly independent list $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \subset \mathbb{R}^2$. Let $u = e_1, v = e_2 \in \mathbb{R}^2$. Then, (u, v) is linearly independent. However, it is not true that

$$\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

is linearly independent.

Whenver you are given a 'prove or give a counterexample' question, ALWAYS search for a counterexample first (in $\mathbb{R}^2$). In my experience, most (probably $4/5$) of these types of questions require you to provide a counterexample. If you can't find a counterexample within 2 minutes then start to think about proving the problem.

5. Let $U, W \subset V$ be subspaces; assume that the sum $U + W$ is direct. Suppose that $(u_1, \dots, u_m) \subset U$ is linearly independent and $(w_1, \dots, w_l) \subset W$ is linearly indpendent. Prove that $(u_1, \dots, u_m, w_1, \dots, w_l)$ is linearly independent.

Solution: Suppose that we have a linear relation

$$a_1 u_1 + \dots + a_m u_m + b_1 w_1 + \dots + b_l w_l = 0_V.$$

We want to show that $a_1 = \dots = a_m = b_1 = \dots = b_l = 0$. Rearranging the above equation we get

$$a_1 u_1 + \dots + a_m u_m = -(b_1 w_1 + \dots + b_l w_l)(= z).$$

The LHS of the above equation is an element of U, the RHS is an element of W. Hence, the element z (which is equal to both the LHS and RHS) must be in both U and W, so that $z \in U \cap W$. Hence, as the sum $U + W$ is direct, we have $U \cap W = \{0_V\}$, and $z = 0_V$. Hence,

$$0_V = a_1 u_1 + \dots + a_m u_m \implies a_1 = a_2 = \dots = a_m = 0 \text{ (since the } u\text{'s are l.i.)},$$

$$0_V = b_1 w_1 + \dots + b_l w_l \implies b_1 = b_2 = \dots = b_l = 0 \text{ (since the } w\text{'s are l.i.)}.$$

The result follows.

6. Let $U \subset \mathbb{C}^3$ be the subspace

$$U = \{x \in \mathbb{C}^3 \mid x_1 + x_2 - x_3 = 0\}.$$

Find a matrix A such that $U = \text{null}(A)$. Find a basis of U (Math 54!).

Suppose now that $W \subset \mathbb{C}^3$ is the subspace

$$W = \{x \in \mathbb{C}^3 \mid 2x_1 - x_3 = 0\}.$$

Find a matrix B such that $U \cap W = \text{null}(B)$ (*Hint: what are the equations defininining $U \cap W$?*) Find a basis of $U \cap W$.

Suppose that $U \subset F^n$ is a subspace defined by k linear equations $L_1 = 0, \dots, L_k = 0$. Find a matrix C such that $U = \text{null}(C)$. How would you find a basis of U?

Solution: We take the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$$

Using the given equation $x_1 + x_2 - x_3 = 0$, we can write any $\underline{x} \in U$ as

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 - x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, the list $\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$ is a spanning list of U. It is easy to show that this list is also linearly independent. Hence, it is a basis of U.

We see that $U \cap W$ is the collection of those vectors in both U and W. Hence, any element of this subspace must satisfy both equations

$$x_1 + x_2 - x_3 = 0, \ 2x_1 - x_3 = 0.$$

Hence, we can take the matrix

$$B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \end{bmatrix}.$$

In order to find a basis of $\text{null}(B)$ we row-reduce B

$$B \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \end{bmatrix},$$

so that $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \text{null}(B)$ if and only if

$$x_1 - \frac{1}{2}x_3 = 0, \ x_2 - \frac{1}{2}x3 = 0,$$

if and only if

$$\underline{x} = x_3 \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

Hence, $U \cap W = \text{span}\left(\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}\right)$.

Suppose that the linear equation $L_i = 0$ is given by

$$a_{i1}x_1 + \dots + a_{in}x_n = 0.$$

Then, we can take the $k \times n$ matrix $C = [c_{ij}]$ (ie, the ij-entry is c_{ij}). To find a basis we row-reduce to reduced echelon form and use the corresponding system of linear equations to eliminate the basic variables (as we've done above). This is Math 54 material.

7. (*In this problem we'll see how we can think of every subspace of F^n as being defined by some linear equations.*)

a) Consider the subspace $U = \text{span}(e_1) \subset F^2$ - find an equation $L = 0$ such that $U = \{x \in F^2 \mid L(x) = 0\}$.

b) Find an equation $L = 0$ such that

$$\text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \{x \in F^2 \mid L(x) = 0\}.$$

c) Can you find an equation $L = 0$ such that

$$U = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right) = \{x \in F^3 \mid L(x) = 0\}?$$

d) Consider the vector $v = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \in F^3$. How can we know when some vector $w \in \text{span}(v)$?

We need to find some $c \in F$ such that $w = cv$; that is, we need to solve the equation $w = cv$, which is possible precisely when the following matrix equation

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} c = w,$$

is consistent. So, if we consider the 'vector of variables' $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ we have

$$[v \mid w] = \begin{bmatrix} -1 & w_1 \\ 1 & w_2 \\ 1 & w_3 \end{bmatrix} \sim \begin{bmatrix} -1 & w_1 \\ 0 & w_1 + w_2 \\ 0 & w_1 + w_3 \end{bmatrix}$$

and in order for the corresponding system of equations to admit a solution we require that

$$w_1 + w_2 = 0, \; w_1 + w_3 = 0.$$

These are the equations we're looking for. Now, apply this method to determine equations defining the following subspaces

$$\operatorname{span}\left(\begin{bmatrix}-1\\2\\3\end{bmatrix}\right),\ \operatorname{span}\left(\begin{bmatrix}1\\0\\-1\\2\end{bmatrix},\begin{bmatrix}0\\4\\1\\2\end{bmatrix}\right),\ \operatorname{span}\left(\begin{bmatrix}-1\\1\\1\\2\end{bmatrix}\right)$$

If I give you $(v_1,\dots,v_m)\subset F^n$, how many equations will you get? (Your answer should contain the word 'pivots').

Solution: a) We can take $U=\{v\in F^2 \mid x_2=0\}$.

b) We can take the equation $x_1-x_2=0$.

c) We see that both vectors defining U satisfy $x_1+x_2+x_3=0$.

d) Consider the matrix

$$\begin{bmatrix}-1 & b_1\\ 2 & b_2\\ 3 & b_3\end{bmatrix}\sim\begin{bmatrix}-1 & b_1\\ 0 & b_2+2b_1\\ 0 & b_3+3b_1\end{bmatrix}$$

Hence, the equations are

$$b_2+2b_1=0,\ b_3+3b_1=0.$$

Consider the matrix

$$\begin{bmatrix}1 & 0 & b_1\\ 0 & 4 & b_2\\ -1 & 1 & b_3\\ 2 & 2 & b_4\end{bmatrix}\sim\begin{bmatrix}1 & 0 & b_1\\ 0 & 4 & b_2\\ 0 & 0 & 4(b_3+b_1)-b_2\\ 0 & 0 & 2(b_4-2b_1)-b_2\end{bmatrix}$$

so that the equations are

$$4b_3+4b_1-b_2=0,\ 2b_4-2b_1-b_2=0.$$

Consider the matrix

$$\begin{bmatrix}-1 & b_1\\ 1 & b_2\\ 1 & b_3\\ 2 & b_4\end{bmatrix}\sim\begin{bmatrix}-1 & b_1\\ 0 & b_2+b_1\\ 0 & b_3+b_1\\ b_4+2b_1 & \end{bmatrix}$$

so that the equations are

$$b_2+b_1=0,\ b_3+b_1=0,\ b_4+2b_1=0.$$

In general, you will get $n-k$ equations, where k is the number of pivots in the matrix $[v_1\cdots v_m]$.