Worksheet 2/5. Math 110, Spring 2014.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Linear independence, span, bases

1. Determine which of the following lists $(v_1, \dots, v_m)$ in $V = F^n$ are

a) linearly independent, b) spanning lists (of V).

Also, say whether the lists can be a basis of V (with appropriate justification).
If the list is not linear independent find a nontrivial linear relation among $(v_1, \dots, v_m)$.

i) $V = \mathbb{R}^3$,

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

ii) $V = \mathbb{C}^3$

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

(Do you need to do anything new here?)

iii) $V = \mathbb{C}^4$

$$v_1 = \begin{bmatrix} \sqrt{-1} \\ 0 \\ 1 \\ -1 \end{bmatrix}, \ v_2 = \begin{bmatrix} -2 \\ 1 \\ \sqrt{-1} \\ 1+\sqrt{-1} \end{bmatrix}, \ v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

iv) $V = \mathbb{C}^2$ (considered as a vector space over $\mathbb{R}$... only allowed $\mathbb{R}$ scalars)

$$v_1 = \begin{bmatrix} \sqrt{-1} \\ -\sqrt{-2} \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}, \ v_3 = \begin{bmatrix} 0 \\ -4 \end{bmatrix}.$$

2. For the last example in Q1, why are you not allowed to use the following 'fact'?

if $m > \dim V$ *then* $(v_1, \dots, v_m)$ *is linearly dependent.*

3. Suppose that $(v_1, \dots, v_m) \subset V$ is linearly independent. Prove that $(v_1, \dots, v_k)$ is linearly independent for any $1 \leq k \leq m$. Can you generalise this to show that any sublist $(v_{i_1}, \dots, v_{i_k})$ is linearly independent? (Hint: yes, you can! But can *you* do it?)

Is it true that if $(v_1, \dots, v_m) \subset V$ is linearly dependent then any sublist is linearly dependent? (*Hint: think of some examples of (nontrivial) linearly dependent sets you know.*)

4. Prove or give a counterexample: if $(v_1, \dots, v_m) \subset V$ is linearly independent and (u, v) is linearly indpendent, with $u, v \notin \operatorname{span}(v_1, \dots, v_m)$, then $(v_1, \dots, v_m, u, v)$ is linearly independent.

5. Let $U, W \subset V$ be subspaces; assume that the sum $U + W$ is direct. Suppose that $(u_1, \dots, u_m) \subset U$ is linearly independent and $(w_1, \dots, w_l) \subset W$ is linearly indpendent. Prove that $(u_1, \dots, u_m, w_1, \dots, w_l)$ is linearly independent.

6. Let $U \subset \mathbb{C}^3$ be the subspace

$$U = \{x \in \mathbb{C}^3 \mid x_1 + x_2 - x_3 = 0\}.$$

Find a matrix A such that $U = \text{null}(A)$. Find a basis of U (Math 54!).

Suppose now that $W \subset \mathbb{C}^3$ is the subspace

$$W = \{x \in \mathbb{C}^3 \mid 2x_1 - x_3 = 0\}.$$

Find a matrix B such that $U \cap W = \text{null}(B)$ (*Hint: what are the equations definining $U \cap W$?*) Find a basis of $U \cap W$.

Suppose that $U \subset F^n$ is a subspace defined by k linear equations $L_1 = 0, \dots, L_k = 0$. Find a matrix C such that $U = \text{null}(C)$. How would you find a basis of U?

7. (*In this problem we'll see how we can think of every subspace of F^n as being defined by some linear equations.*)

a) Consider the subspace $U = \text{span}(e_1) \subset F^2$ - find an equation $L = 0$ such that $U = \{x \in F^2 \mid L(x) = 0\}$.

b) Find an equation $L = 0$ such that

$$\text{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \{x \in F^2 \mid L(x) = 0\}.$$

c) Can you find an equation $L = 0$ such that

$$U = \text{span}\left(\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right) = \{x \in F^3 \mid L(x) = 0\}?$$

d) Consider the vector $v = \begin{bmatrix}-1\\1\\1\end{bmatrix} \in F^3$. How can we know when some vector $w \in \text{span}(v)$?

We need to find some $c \in F$ such that $w = cv$; that is, we need to solve the equation $w = cv$, which is possible precisely when the following matrix equation

$$\begin{bmatrix}-1\\1\\1\end{bmatrix} c = w,$$

is consistent. So, if we consider the 'vector of variables' $w = \begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}$ we have

$$[v \mid w] = \begin{bmatrix}-1 & w_1\\1 & w_2\\1 & w_3\end{bmatrix} \sim \begin{bmatrix}-1 & w_1\\0 & w_1 + w_2\\0 & w_1 + w_3\end{bmatrix}$$

and in order for the corresponding system of equations to admit a solution we require that

$$w_1 + w_2 = 0, \; w_1 + w_3 = 0.$$

These are the equations we're looking for. Now, apply this method to determine equations defining the following subspaces

$$\operatorname{span}\left(\begin{bmatrix}-1\\2\\3\end{bmatrix}\right), \operatorname{span}\left(\begin{bmatrix}1\\0\\-1\\2\end{bmatrix}, \begin{bmatrix}0\\4\\1\\2\end{bmatrix}\right), \operatorname{span}\left(\begin{bmatrix}-1\\1\\1\\2\end{bmatrix}\right)$$

If I give you $(v_1, \ldots, v_m) \subset F^n$, how many equations will you get? (Your answer should contain the word 'pivots').