Worksheet 1/29. Math 110, Spring 2014.

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Sums; direct sums

1. Consider the following sums $U + W \subset V$, where $U, W \subset V$ are subspaces (check!). Which of the sums $U+W$ are direct sums? If they are not direct sums find a nonzero vector $v \in U \cap W$ and write an element $z \in U + W$ as $z = u + w = u' + w'$, with $u \neq u'$, $w \neq w'$.

i) $V = \mathbb{R}^3$

$$U = \{\underline{x} \in V \mid x_1 + x_2 + x_3 = 0\},\ W = \{\underline{x} \in V \mid x_1 - x_2 = x_3\}.$$

ii) $V = \mathbb{R}^3$,

$$U = \{(t, 0, -t) \mid t \in \mathbb{R}\},\ W = \{\underline{x} \in V \mid 2x_2 + x_3 = 0\}.$$

iii) $V = \mathbb{C}^3$,

$$U = \{(t, t, t) \mid t \in \mathbb{C}\},\ W = \{(2u, -u, u) \mid u \in \mathbb{C}\}.$$

iv) $V = Mat_2(\mathbb{R})$,

$$U = \{A \in V \mid AB = 0\},\ W = \{A = [a_{ij}] \in V \mid a_{11} + a_{22} = 0\},$$

where $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and an arbitrary 2×2 matrix A is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Solution:

i) We will use the criterion that $U + W$ is a direct sum if and only if $U \cap W \neq \{0\}$. So, we are looking to determine what the vector space $U \cap W$ is. Note that

$$\begin{aligned} U \cap W &= \{\underline{x} \in V \mid \underline{x} \in U \text{ and } \underline{x} \in W\} \\ &= \{\underline{x} \in V \mid x_1 + x_2 + x_3 = 0, x_1 - x_2 - x_3 = 0\} \\ &= nul\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}\right) \end{aligned}$$

and since the columns of the matrix given above are not linearly independent then the nullspace is nontrivial (there is not a pivot in each column - MATH 54!!). Hence, there is a nontrivial (= nonzero) vector in $U \cap W$ so that the sum can't be direct. Recalling Math 54 methods in determining vectors in nullspaces (ie, row-reduction) we see that the vector

$$v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \in U \cap W.$$

In fact, $U \cap W = \text{span}(v)$. Moreover, we see that we can write

$$v = v + 0_V = 0_V + v,$$

with $u = v, w = 0_V,\ u' = 0_V, w' = v$.

ii) A vector $v \in V$ is an element of $U \cap W$ if there is a $t \in \mathbb{R}$ such that

$$v = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} \left(= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)$$

and where we further require that $0 = 2x_2 + x_3 = 0 - t \implies t = 0$. Hence, $v = \underline{0}$ and $U \cap W = \{0\}$, so that $U + W$ is a direct sum.

iii) Proceeding above, a vector $v \in U \cap W$ must be of the form

$$v = \begin{bmatrix} t \\ t \\ t \end{bmatrix}, \quad \text{for some } t \in \mathbb{C},$$

and it must also be of the form

$$v = \begin{bmatrix} 2u \\ -u \\ u \end{bmatrix}, \quad \text{for some } u \in \mathbb{C}.$$

Hence, we must have $u, t \in \mathbb{C}$ such that

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 2u \\ -u \\ u \end{bmatrix} \implies u = t, \quad \text{and } u = -t.$$

The only possibility is that $u = t = 0$ so that $v = \underline{0}$. Hence, $U \cap W = \{0\}$ and the sum is direct.

iv) If a matrix $A \in W$ then we must have

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}.$$

Furthermore, we require that

$$0 = AB = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}.$$

Hence, we must have that $a = c = 0$, but there is no restriction on b. For example, we see that if we take $b = 1$ then the matrix B is an element of $U \cap W$. Hence, $U \cap W \neq \{0\}$ and the sum is not direct.

Similarly as above, we can take $u = B, w = 0$ and $u' = 0, w' = B$.

2. Consider the subspace

$$U = \{\underline{x} \in \mathbb{R}^5 \mid x_1 + x_2 + x_3 + x_4 + x_5 = 0\} \subset \mathbb{R}^5$$

Find three distinct subspaces $W_1, W_2, W_3 \subset \mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W_i$ $(i = 1, 2, 3)$, making sure that you justify how you know that the W_i you have chosen makes the sum $U + W_i$ direct.

Are there a finite number of subspaces $W \subset \mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$? Justify your answer.

Solution: Since

$$\underline{x} \in U \Leftrightarrow x_1 + x_2 + x_3 + x_4 + x_5 = 0 \Leftrightarrow x_1 = -x_2 - x_3 - x_4 - x_5,$$

then we must have

$$\underline{x} = \begin{bmatrix} -x_2 - x_3 - x_4 - x_5 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

If we call the vectors on the RHS of the above equation v_1, v_2, v_3, v_4 (respectively), then we have just shown that

$$U = \text{span}(v_1, v_2, v_3, v_4).$$

Moreover, the 5×4 matrix whose columns are v_1, v_2, v_3, v_4 has a pivot in each column so that these columns are linearly independent. Hence, $(v_1, \dots, v_4)$ is a basis of U.

Our strategy is to find a vector $w \notin U$ - then, $W = \text{span}(w)$ is a subspace such that $U \oplus W = \mathbb{R}^5$. Let's take $w_1 = e_1, w_2 = e_2, w_3 = e_3$ (where e_i is the i^{th} standard basis vector). Then, if we define $W_i = \text{span}(w_i)$ we have that $U \cap W_i = \{0\}$ - this is because no nonzero scalar multiple of w_i is an element of U (can you write this down?). Moreover, we have that $U + W_i = \mathbb{R}^5$ since the matrix

$$[v_1 \ v_2 \ v_3 \ v_4 \ w_i]$$

has a pivot in every row, so that the columns span $\mathbb{R}^5$. This is precisely the same thing as saying that $U + W_i = \mathbb{R}^5$.

There are infinitely many subspaces $W \subset \mathbb{R}^5$ such that $U \oplus W = \mathbb{R}^5$ since all that we need to do is choose some vector $w \notin U$ and then $W = \text{span}(w)$ satisfies the reuquired conditions. To see there are infinitely many choices we can take

$$w = \begin{bmatrix} n \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{for } n \text{ a positive integer.}$$

3. Suppose that $U \subset \mathbb{R}^3$ is a subspace of dimension 2 (recall the notion of dimension from Math 54). Give a criterion to find a subspace $W \subset \mathbb{R}^3$ such that $\mathbb{R}^3 = U \oplus W$.

Suppose now that $U \subset \mathbb{R}^3$ is a subspace of dimension 1. Can you see how to give a criterion to find subspaces $W, R \subset \mathbb{R}^3$ such that $\mathbb{R}^3 = U \oplus W \oplus R$. What should the dimension of W, R be?

Solution: Choose $w \notin U$, $w \neq 0$. Then, $W = \text{span}(w)$ will satisfy $U \oplus W = \mathbb{R}^3$.

If U has dimension one, say $U = \text{span}(u)$ with $u \neq 0$, then we choose $w \notin U$ and let $W = \text{span}(w)$. Then, the sum $U + W$ is direct. Now, we take $r \notin (U + W)$ and let $R = \text{span}(r)$, so that the sum $(U + W) + R$ is direct. Moreover, the list (u, w, r) is a basis of $\mathbb{R}^3$ (eg it's a linearly independent list since, if there was a linear dependence relation among (u, w, r) then we'd break the assumptions on $w \notin U$ and $r \notin U + W$; since it's a linearly independent list of length 3 in a three dimensional space it must be a basis). This implies that $\mathbb{R}^3 = U \oplus W \oplus R$ (use the definition of what it means for a sum of several subspaces to be a direct sum).

4. (*This is for those of you that recall the notion of a basis of $\mathbb{R}^3$. If you can't remember the definition then try this problem again in a week or so...*) Can you use the previous problem to show how to find a basis of $\mathbb{R}^3$, containing $u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \in \mathbb{R}^3$?

Solution: The approach was outlined in the problem above: take $w \notin U = \text{span}(u)$, for example, we can take $w = e_1$. Then,

$$U + W = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}.$$

So, if we take $r = e_3$ then $r \notin U + W$ and you can check that the list (u, w, r) is a basis (Eg consider the matrix $[u\ w\ r]$ and show that there is a pivot in each column/row).