Worksheet 1/22. Math 110, Spring 2014. Solutions

These problems are intended as supplementary material to the homework exercises and will hopefully give you some more practice with actual examples. In particular, they may be easier/harder than homework. Send me an email if you have any questions!

Vector spaces; subspaces

1. Which of the following sets V are vector spaces over $\mathbb{R}$ (**Note: this is changed from the worksheet handed out in discussion.**), with the given vector addition and scalar multiplication? You will need to check <u>all</u> of the axioms! (Sorry...)

i) $V = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\} \subset \mathbb{R}^3$, with the 'usual' addition of vectors and scalar multiplication inherited from $\mathbb{R}^3$.

ii) $V = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = \pi\} \subset R^3$, with the 'usual' addition of vectors and scalar multiplication inherited from $\mathbb{R}^3$.

iii) Let $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and

$$V = \{2 \times 2 \text{ matrices } A \text{ with real entries } \mid AB = 0 \text{ (the zero } 2 \times 2 \text{ matrix)}\} \subset Mat_2(\mathbb{R}),$$

with the 'usual' addition of vectors and scalar multiplication inherited from $Mat_2(\mathbb{R})$, the set of all 2×2 matrices with real entries.

iv) (*Tricky!*) $V = \{(1, y, z) \in \mathbb{R}^3\} \subset \mathbb{R}^3$, where we define vector addition as

$$(1, y, z) + (1, u, v) = (1, y + u, z + v),$$

and scalar multiplication as

$$c \cdot (1, y, z) = (1, cy, cz), \; c \in \mathbb{R}.$$

Note: For this example you will need to say what an appropriate *zero vector* in V should be.

Solution:

i) First we need to check whether the notions of addition and scalar multiplication are well-defined - that is, if I give you any two vectors $u, v \in V$ is there sum $u + v$ also in V, and if $c \in \mathbb{R}$ is $cv \in V$? Well, let $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in V$. Thus, we know that

$$u_1 + u_2 + u_3 = 0, \; v_1 + v_2 + v_3 = 0.$$

Now, $u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$ and we need to check if the coordinates sum to 0 to show that $u + v \in V$. Indeed,

$$(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) = (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3) = 0 + 0 = 0.$$

Hence, we see that $u + v \in V$ and our notion of addition that we've defined in V is well-defined. Also, we have $cv = (cv_1, cv_2, cv_3)$ and

$$cv_1 + cv_2 + cv_3 = c(v_1 + v_2 + v_3) = c.0 = 0.$$

Hence, $cv \in V$ so our notion of scalar multiplication is well-defined.

Commutativity and associativity of addition have already been verified in Axler's book (since they are inherited from $\mathbb{R}^3$).

(additive identity) the obvious choice is the zero vector in $\mathbb{R}^3$ (it is indeed an element in V!). Since every $v \in V$ is also a vector in $\mathbb{R}^3$ (because $V \subset \mathbb{R}^3$) then we know that, for every $v \in V \subset \mathbb{R}^3$, we have $v + 0_V = v$, as the additive identity axiom holds in $\mathbb{R}^3$. Hence, the additive identity axiom hols for V also. (This kind of seems like we've done nothing here, but we have! Make sure you can appreciate the previous argument)

(additive inverse) let $v = (x, y, z) \in V$. We need to show that there is $w \in V$ such that $v + w = 0_V$, where 0_V is the zero vector. Of course, the obvious candidate for w is the vector $(-x, -y, -z)$; however, we need to check that $w \in V$: well, obviously we have $(-x) + (-y) + (-z) = -(x + y + z) = 0$, since we have assumed that $v \in V$ so that $x + y + z = 0$. Hence, to any $v \in V$ there is an additive inverse.

Again, the properties for scalar multiplication (multiplicative identity and distributive properties) hold for V because they are inherited from $\mathbb{R}^3$.

ii) Let's check to see if our notion of addition is well-defined: let $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in V$, so that

$$u_1 + u_2 + u_3 = \pi,\ v_1 + v_2 + v_3 = \pi.$$

Then, we have $u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$ and we need to check if the coordinates sum to π, so that $u + v \in V$. However, now we have

$$(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) = (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3) = \pi + \pi = 2\pi \neq \pi.$$

Hence, $u + v \notin V$ and our notion of addition is not well-defined. Thus, V with the given definitions of addition and scalar multiplication is not a vector space (over $\mathbb{R}$).

iii) For this example we need to recall some basis properties of matrix arithmetic (from Math 54): if X, Y, Z are 2×2 matrices with real entries, $c \in \mathbb{R}$, then

$$(X + Y)Z = XZ + YZ,\ (cX)Y = c(XY).$$

As above we need to check that the notions of addition and scalar multiplication are well-defined. Let $X, Y \in V$ be arbitrary, so $XB = 0, YB = 0$ - we want to show that $X + Y \in V$, so that the notion of addition we have on V is well-defined. In order that $X + Y \in V$ we must have that $X + Y$ satisfies the condition defining the set V, namely, we need that $(X + Y)B = 0$. Recall our basic matrix arithmetic we find

$$(X + Y)B = XB + YB = 0 + 0 = 0.$$

Hence, $X + Y \in V$ and addition is well-defined. Similarly, we must show that scalar multiplication is well-defined. Let $c \in \mathbb{R}$. Then, we need to check if $cX \in V$. Indeed,

$$(cX)B = c(XB) = c0 = 0.$$

Hence, we have that $cX \in V$ and scalar multiplication is well-defined.

Commutativity and associativity of addition hold since addition is inherited from $Mat_2(\mathbb{R})$, which is a vector space (over $\mathbb{R}$) (this should be Math 54 stuff). Similarly, the multiplicative identity and distributive properties axioms hold.

(additive identity) the obvious choice is the 2×2 zero matrix, which we'll denote 0_V. We must check that $0_V \in V$ first: indeed, we have

$$0_V B = 0,$$

so that $0_V \in V$. Also, because 0_V is the additive identity in $Mat_2(\mathbb{R})$ then, for any $X \in Mat_2(\mathbb{R})$, we have $X + 0_V = X$. In particular, for any $X \in V$ we have $X + 0_V = X$. Hence, the additive identity axiom holds.

(additive inverse) this is similar to (i) - you just need to check that the additive inverse you come up with is actually an element of V.

iv) First note that the addition and scalar multiplication are well-defined (in the same sense as above).

(commutativity) let $u = (1, x, y), v = (1, z, w) \in V$. Then, we have

$$u+v = (1, x, y)+(1, z, w) = (1, x+z, y+w) = (1, z+x, w+y) = (1, z, w)+(1, x, y) = v+u.$$

The associativity axiom is similar. Can you see what to do for the multiplicative identity and distributive properties axioms? (They should be straightforward, hopefully!)

Defining $0_V = (1, 0, 0)$ you verify that this gives an additive identity for V, and that for $u = (1, x, y)$ an additive inverse is $w = (1, -x, -y) \in V$. Hence, V is a vector space over $\mathbb{R}$.

2. All of the above sets are given as subsets of another vector space (which one?). Which of these subsets are subspaces? For those subsets that you think are subspaces prove that they are, in fact, subspaces.

Solution: Only *i*) and *iii*) are subspaces - the other two do not contain the zero vectors in the larger vector spaces. To prove that *i*) and *iii*) are indeed subspaces (of $\mathbb{R}^3$ and $Mat_2(\mathbb{R})$ respectively) you need to verify the three axioms

- $0_{\mathbb{R}^3} \in V$
- if $u, v \in V$ then $u + v \in V$ (where the '$+$' is the addition defined in the larger vector space)
- if $u \in V$, $c \in \mathbb{R}$, then $c \cdot u \in V$ (where '$\cdot$' is the scalar multiplication defined in the larger vector space).

We actually verified these conditions when we were checking that the notions of addition and scalar multiplication were well-defined.

3. Consider the set

$$V = \{e^c \mid c \in \mathbb{R}\} = \mathbb{R}_{>0},$$

and define an 'addition'

$$e^c \oplus e^d \overset{def}{=} e^{c+d},$$

and a 'scalar multiplication'

$$\lambda \cdot e^c \overset{def}{=} e^{\lambda c}.$$

Is V a real vector space when we define addition and scalar multiplication in this way? Prove or explain why not.

Solution: The notions of addition and scalar multiplication are well-defined (the intended 'sums' and 'scalar multiples' are elements of V). Moreover, commutativity and associativity of addition are straightforward to verify (if you'd like for me to do this then just ask!).

(additive identity) let's try $0_V = e^0 \in \mathbb{R}_{>0}$. Then, for any $u = e^c \in V$ we have

$$u \oplus 0_V = e^c \oplus e^0 = e^{c+0} = e^c = u.$$

Hence, $0_V = 1$ is an additive identity.

(additive inverse) for $u = e^c \in V$, we take $w = e^{-c} \in V$ - then $u \oplus w = 0_V$.

(mutliplicative identity) let $u = e^c \in V$. Then, we have

$$1 \cdot u = 1 \cdot e^c = e^{1c} = e^c = u.$$

Thus, this axiom is verified.

(distributive properties) let $u = e^c, v = e^d \in V$, $a, b \in \mathbb{R}$. Then,

$$a \cdot (u \oplus v) = a \cdot (e^c \oplus e^d) = a \cdot (e^{c+d}) = e^{a(c+d)} = e^{ac+ad} = e^{ac} \oplus e^{ad} = (a \cdot e^c) \oplus (a \cdot e^d) = (a \cdot u) \oplus (a \cdot v).$$

The other property is similar.