· Koszul algebras and Koszul duality:

- k - field
- V/k - f.d. vector space.
- $R \subseteq V \otimes V$.

} "quadratic data"

$\rightsquigarrow \quad A = T(V)/\langle R \rangle$ "quadratic ring"

eg · $R = \mathrm{Sym}^2(V) \Rightarrow A = \Lambda(V)$
$R = \mathrm{Alt}^2(V) \Rightarrow A = S(V)$

· Denote $R^* = \{ f \in V^* \otimes V^* \mid f(r) = 0, \forall r \in R \} \subseteq V^* \otimes V^*$

Define: $A^! = T(V^*)/\langle R^* \rangle$ "quadratic dual of A"

eg: $S(V)^! \cong \Lambda(V^*)$
$\Lambda(V)^! \cong S(V^*)$
$\rightsquigarrow (-)^!$ involutive in this case.

· Would like to understand cohomology for various functors out of A-mod etc.

· Koszul ~~(resolution)~~ complex: $A = \bigoplus_{i \geq 0} A_i, \quad A_0 = k.$
$A \to k$, want to resolve $_A k$ by free A-mods

$$A \otimes (V^{\otimes 2} \otimes R \cap V \otimes R \otimes V \cap R \otimes V^{\otimes 2}) \to A \otimes (R \otimes V \cap V \otimes R) \to A \otimes R \to A \otimes_k V \to A \to k$$

$a \otimes v \mapsto a \cdot v$

ker $= A_{>0}$

- "<u>Def</u>^{n}": A is <u>Koszul</u> if the Koszul complex is a resolution of k.

<u>eg</u>: $S(V)$ is Koszul:

$$\cdots \longrightarrow S(V)\otimes \Lambda^3 V \longrightarrow S(V)\otimes \Lambda^2 V \longrightarrow S(V) \longrightarrow k$$ is a resolution.

("classical" Koszul complex)

- $$\bigcap_{\nu} V^{\otimes \nu} \otimes R \otimes V^{(i-\nu-2)} \simeq (A^!_i)^* \simeq \left(\frac{(V^*)^{\otimes i}}{\langle \text{stuff gen'd by } R^* \rangle} \right)^*$$

$\rightsquigarrow$ Koszul complex:

$$\cdots \longrightarrow A\otimes (A^!_2)^* \longrightarrow A\otimes (A^!_1)^* \longrightarrow A\otimes (A^!_0)^* \longrightarrow k \qquad (K_\bullet)$$

obtain a double complex.

$$\longrightarrow A_1\otimes (A^!_1)^* \longrightarrow A_1\otimes (A^!_0)^*$$

$$\longrightarrow A_0\otimes (A^!_1)^* \longrightarrow A_0\otimes (A^!_0)^* \longrightarrow k$$

$\rightsquigarrow$ If $K_\bullet$ is a resolution then cohom. is concentrated in bottom right; computing the cohomology of this bicomplex in 2 ways gives

<u>Prop</u>: If A Koszul then $A^!$ Koszul. "Koszul dual"

Prop: $\mathrm{Ext}^{\bullet}_A(k,k) \simeq (A^!)^{op}$

- **Morita theory for rings:**

Let R, S be rings.

Question: When is $R\text{-mod} \simeq S\text{-mod}$?

distinguished

${}_R R$

- $(M \longmapsto \mathrm{Hom}_R(R,M))$ is faithful functor.
- $R^I \longrightarrow R^J \longrightarrow M \longrightarrow 0$

for such R, R is a "generator"

- $\mathrm{Hom}_R(R,-)$ – preserves (infinite) coproducts (R small/compact)

– exact (R projective)

- **Theorem:**

$\mathcal{A}$ be abelian category (w/ ∞ coproducts), P small projective generator.

Then,
$$\mathrm{Hom}(P,-) : \mathcal{A} \xrightarrow{\sim} \text{mod} - \mathrm{End}_{\mathcal{A}}(P,P).$$
is an equivalence of categories.

- **Corollary:** $R\text{-mod} \simeq S\text{-mod}$

$\iff$ R-mod contains a small proj. generator P with $\mathrm{End}(P) \simeq S^{op}$

eg: $R^n \in R\text{-mod}$

$\mathrm{End}(R) \simeq \mathrm{Mat}_n(R^{op})$

$\Rightarrow \quad R\text{-mod} \simeq \mathrm{Mat}_n(R)\text{-mod}.$

- <u>Derived Morita Theory:</u>

<u>Q^n</u>: When are $D^?(R) \simeq D^?(S)$? (as triangulated categories)

<u>Recall:</u> $\mathcal{A}$ - abelian category

$\rightsquigarrow$

$Ch(\mathcal{A})$ - chain complexes in $\mathcal{A}$.

$\rightsquigarrow$

$K(\mathcal{A})$ - homotopy category

$\rightsquigarrow$

$D(\mathcal{A})$ - "formally invert qis"

- <u>Theorem</u>: R, S are derived Morita equivalent

$\Leftrightarrow$ $D(R)$ contains a <u>tilting complex</u> T with

$$R\,\mathrm{End}(T) \simeq S^{op},$$

and equivalence given by $R\,\mathrm{Hom}(T,-)$.

tilting complex $\Leftrightarrow$
- qis to bounded cx of f.g. projective modules ("small")
- coprods, cokernels, shifts, Δ's ("generator")

Theorem: A graded ring, $A = \bigoplus_{i \geq 0} A_i$. $\dim A < \infty$

Then, $R\mathrm{Hom}(k, -)$ gives a derived Morita equivalence from

$$D^b(A\text{-}\underset{\substack{\uparrow \\ \text{f.g.}}}{\text{gmod}}) \longrightarrow D^b(A^!\text{-}\underset{\substack{\uparrow \\ \text{f.g.}}}{\text{gmod}})$$

We call $K = R\mathrm{Hom}(k, -)$ the <u>Koszul duality functor</u>.

$\rightsquigarrow$ BGG correspondence. (next week)
