KOSZUL SEMINAR: PABLO

10/31

①

Ref: Braden - "Koszul Duality for toric varieties"

Outline:

⓪ Fans and stuff

① Perverse sheaves

② Main theorem

③ Sketch of proof

④ (some) details.

"Combinatorics is like candy... some is ok, too much gives you stomach ache"

Combinatorics:

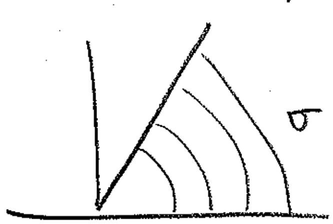
$L \simeq \mathbb{Z}^n$ | $L^* = \mathrm{Hom}(L, \mathbb{Z})$

$V = L \otimes \mathbb{R}$ | $V^* = L^* \otimes \mathbb{R}$

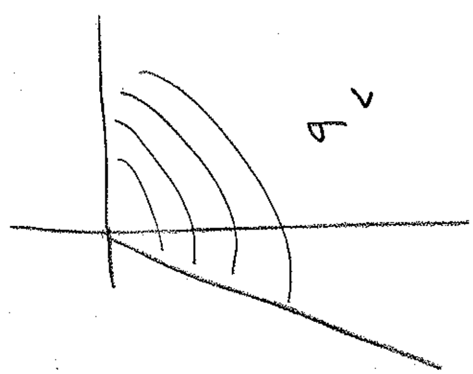
cone $\sigma \subseteq V$ | $\sigma^\vee = \{ f \mid f(\sigma) \geq 0 \} \subseteq V^*$.

slope = 2

eg



σ



$\sigma^\vee$

slope = $-1/2$.

$\tau \leq \sigma$
subcone

$$V_\tau^\perp = \{ f \in V^* \mid f(\tau) = 0 \}$$

$$\tau^\perp = V_\tau^\perp \cap \sigma^\vee$$

$$\tau \xleftrightarrow[\text{bijection}]{\text{inc. reversing}} \tau^\perp$$

cones in σ $\longleftrightarrow$ cones in $\sigma^\vee$

Geometry:

$X(\sigma)$ - variety$/\mathbb{C}$	$X(\sigma^\vee)$
$\cup\!\!\!\mid$	''
$T = \hom(L^*, \mathbb{C}^\times)$	''
$T \times X(\sigma) \longrightarrow X(\sigma)$	''
	''
$\rightsquigarrow$ orbit stratification of $X(\sigma)$	''

Topology

Cone σ, fan $[\sigma] = \{ \tau \leq \sigma \}$.

Topologise: $[\sigma]$; open basis $[\tau]$, $\tau \leq \sigma$.

Λ_σ sheaf of rings on $[\sigma]$

$$[\tau] \longrightarrow \Lambda^*(V_\tau^\perp)$$

- ($X(\sigma)$, orbit stratification)

$$\rightsquigarrow D^b(X(\sigma)) \text{ , with t-structure}$$

$$\rightsquigarrow P(X(\sigma)) = \text{heart}$$

$$[\sigma] \rightsquigarrow D(\Lambda_\sigma\text{-mod}), \text{ t-structure}$$

$$\rightsquigarrow \mathbb{P}(\sigma) = \text{heart}(\mathcal{D}(\sigma), t)$$

- To bring in Koszul rings we need to restrict to nice subcategories that depend on Φ, inner product on V

$$P_\Phi(X(\sigma)) \qquad\qquad \mathbb{P}_\Phi(\sigma)$$

simple : $IC(O_\tau^+, \mathbb{R}) \overset{\text{def}}{=} L_\tau$

- all objects have finite length
- there is an inj. cogenerator $I = \bigoplus I_\tau$

inj. hull of L_τ.

General hom. alg

$$\Rightarrow P_\Phi(X(\sigma))^{op} \simeq A\text{ - fin gen. mod.}$$

$$A = \mathrm{End}_{P_\Phi}(I)$$

$$A^! = \mathrm{End}_{\mathbb{P}_{\Phi^\vee}}(I^\vee)$$

Theorem: (MAIN 1) $\|$ A, $A^!$ are Koszul dual.

<u>Theorem</u>: (Main 1)

$$(A^!)^{opp} \simeq \mathrm{Ext}^\bullet_\Phi(L, L), \quad L = \bigoplus L_\tau$$

$\uparrow$ This is Koszul.

- Main 1 $\Leftarrow$ <u>Thm (Main 2)</u>:

There is an equivalence

$$K : D(A\text{-mod}_{gr.})^{opp} \longrightarrow D(A^!\text{-mod})$$

s.t.

a) $K(M\langle n\rangle) = (KM)[-n]\langle -n\rangle$ — hom. shift; grading shift

b) $K(L_\tau) \simeq I_{\tau^\perp}$, $K(I_\tau) \simeq L_{\tau^\perp}$.

- <u>Construction of K:</u>

$$\begin{tikzcd}
D^b(A\text{-mod})^{opp} \arrow[d, phantom, "\wr\wr"] & & D^b(A^!\text{-mod}) \arrow[d, phantom, "\wr\wr"] \\
D^b(P_\Phi(X(\sigma))) \arrow[d, phantom, "\wr\wr \text{ hard part}"] & & (-)^\vee \wr\wr (-)^\vee \arrow[d, phantom, "\wr\wr"] \\
D^b(P_\Phi(\sigma)) \arrow[r, phantom, "\wr\wr"] & D_\Phi(\sigma) \arrow[r, "K"] & D_\Phi(\sigma^\vee)
\end{tikzcd}$$

- K is derived functor of

$$K : \Lambda_\sigma\text{-mod} \longrightarrow \Lambda_{\sigma^\vee}\text{-mod}.$$

costd objects $N_\tau \in D(\sigma)$ preserved by K

$$K(N_\tau) = N_{\tau^\perp}$$

① There is an inductive alg. way to construct L_τ from N_τ

② " " I_τ from N_τ.

- K interchanges ① and ②

- <u>Remark on combinatorics</u>:

$$\begin{array}{ccccc} \mathrm{Lie}(\mathcal{O}_\tau) & \xrightarrow{\exp} & (\mathbb{C}^\times)^{\mathrm{codim}(\tau)} & \subseteq & X(\sigma) \\ \| & & \wr\| & & \\ V/\mathrm{span}(\tau) & & \mathcal{O}_\tau & & \end{array}$$

- $\Lambda_\tau([\tau]) = \Lambda^\cdot(V_\tau^\perp)$ const. diff. forms on $\mathrm{Lie}(\mathcal{O}_\tau)$

- $\Lambda_\tau \longrightarrow \Gamma(\mathcal{O}_\tau, \Omega^*(\mathbb{R}))$, $\tau \overset{i_\tau}{\leq} \sigma$

$i_{\tau *}\Lambda_\tau$ sheaf on $[\sigma]$ $\rightsquigarrow$ resolves $\mathbb{R}_{\mathcal{O}_\tau}$

- Define $\hom^{\oplus}_{D(A^1\text{-mod})}(M,N) = \bigoplus \hom(M,N)$

Main 2 $\Rightarrow$ $K(\hom^{\oplus}(L,L)) \simeq \hom(L,L)$

$$\mathrm{End}(I^\vee) = \hom_D(I^\vee, I^\vee) \qquad\qquad \hom(L,L) = \mathrm{Ext}^*(L,L)$$