Koszul Seminar: Gunnar (1) 10/24/13

Ref: • Eisenbud, Fløystad, Schreyer: "Sheaf cohom..." (2003)
• Beilinson, Ginzburg, Soergel: "Koszul duality patterns..."
• Bernstein, Gelfand, Gelfand: "Algebraic vector bundles..."
• Macaulay 2

. How Tate resolutions compute cohomology of coherent sheaves on projective spaces.

1. Setup of Koszul duality:

$\deg V = 1$
$\deg V^* = -1$

$V = k\{x_1, \ldots, x_n\}$, $V^* = k\{e_1, \ldots, e_n\}$, $\langle e_i, x_j \rangle = \delta_{ij}$

$R \subseteq V \otimes_k V \rightsquigarrow A = T(V)/(R)$ "Koszul algebra."

↑ two sided ideal

i.e. resolution is linear.

$\rightsquigarrow R^\perp \subseteq V^* \otimes V^* \twoheadrightarrow R^*$

$A^! = T(V^*)/(R^\perp)$ "dual Koszul algebra"

• $A^! = \bigoplus_{d \geq 0} A^!_d$ ← $\deg = -d$.

. Consider the 'graded dual' of $A^!$,

$$(A^!)^{\circledast} \underset{\text{def}}{=} \bigoplus_{d \geq 0} (A^!_d)^*$$ ← deg d.

Let N be graded (left) $A^!$-module, and consider

$$a \otimes n \longmapsto \sum a x_i \otimes e_i n$$

$$F(N): \quad \cdots \longrightarrow A(i) \otimes_k N_{-i} \longrightarrow A(i+1) \otimes_k N_{-i-1} \longrightarrow \cdots$$

consider N_{-i} as (ungraded) v.s.
i.e. grading comes from LH term.

- Suppose N is a complex

$$\cdots \longrightarrow N^j \longrightarrow N^{j+1} \longrightarrow \cdots$$

and consider the double complex

$$\begin{tikzcd}
 & \arrow[d] & \\
\cdots \arrow[r] & A(i) \otimes_k N^j_{-i} \arrow[r] \arrow[d] & A(i+1) \otimes N^j_{-i-1} \arrow[r] \arrow[d] & {} \\
\cdots \arrow[r] & A(i) \otimes N^{j+1}_{-i} \arrow[r] \arrow[d] & A(i+1) \otimes N^{j+1}_{-i-1} \arrow[r] \arrow[d] & {} \\
 & \vdots & \vdots &
\end{tikzcd}$$

$\rightsquigarrow \quad F(N) =$ total (direct sum) complex of double cx

- For M a graded (left) A-module.

$$G(M): \quad \cdots \longrightarrow \mathrm{Hom}_k(A^!(i), M_i) \longrightarrow \mathrm{Hom}_k(A^!(i+1), M_{i+1}) \to \cdots$$

graded hom.

$$\| $$

$$(A^!)^{\circledast}(-i) \otimes_k M_i$$

For M cx $\rightsquigarrow$ $G(M)$ total (direct product) cx of double cx.

Adjoint functors

$$\begin{tikzcd} C(A^!) \arrow[r, shift left, "F"] & C(A) \arrow[l, shift left, "G"] \\ A^!\text{-mods} \arrow[u] & A\text{-mods} \arrow[u] \end{tikzcd}$$

$$F(N) = A \otimes_k (A^!) \otimes_A N$$

"adjunction is tensor-hom adjunction"

$$G(M) = \mathrm{Hom}_A(A \otimes_k A^!, M)$$

Tate Resolutions

- Now, $A = \mathrm{Sym}(V) = S(V)$

$$A^! = E(V^*)$$

$$(A^!)^{\circledast} = \hat{E} = \bigoplus_{i=0}^{\dim V} \Lambda^i V$$

Note: $\hat{E} \simeq E$ as left E-module; ie E is "self injective".

- For N graded f.g. (left) E-module.

Free resolution: $\cdots \to T^{-2} \to T^{-1} \to T^0 \to N$ (1)

Cofree (injective) resolution $N \to T^1 \to T^2 \to \cdots$ (2)

where $T^i = \bigoplus_{j \in \mathbb{Z}} \hat{E}(-j)^{\gamma_{ij}}$.

$\rightsquigarrow$ splice together: $\cdots \to T^{-2} \to T^{-1} \to T^0 \to T^1 \to T^2 \to \cdots$

"TATE RESOLUTION"

<u>Properties:</u>

1) Doubly infinite (if N not projective)
2) unique up to htpy
3) Exact everywhere

- Assume (1), (2) are <u>minimal</u> free/cofree resolutions (ie $T^{\bullet} \otimes k$ has trivial differentials)

4) Unique up to isomorphism
5) The whole cx is determined by <u>any</u> of its differentials

- <u>Tate resolution</u>.

Everywhere exact cx of graded free f.g. (left) E-modules.

$\mathrm{Tate}(E)$ = htpy category

$\mathrm{Tate}^{\min}(E)$ = htpy category of min. Tate res^{ns}.

3. <u>Coherent sheaves</u>

Let $\mathcal{F}$ be coherent sheaf on $\mathbb{P}(V) = \mathrm{Proj}(S(V))$

$$\rightsquigarrow \quad M = \bigoplus_{d \in \mathbb{Z}} \underset{\substack{\parallel \\ H^0(\mathbb{P}(V), \mathcal{F}(d))}}{\Gamma(\mathbb{P}(V), \mathcal{F}(d))}, \quad \text{graded } S(V)\text{-module.}$$

Consider $G(M)$:

$$\begin{array}{c} \cdots \longrightarrow \hat{E}(-d) \otimes_k H^0(\mathbb{P}(V), \mathcal{F}(d)) \\ \downarrow \\ \hat{E}(-d-1) \otimes_k H^0(\mathbb{P}(V), \mathcal{F}(d+1)). \\ \downarrow \\ \vdots \end{array}$$

This sequence is exact for some $d \longrightarrow \cdots$

$\rightsquigarrow$ obtain a Tate resolution $T(\mathcal{F})$

(find d where exactness holds, consider image and take projective resolution)

<u>Theorem:</u> (EFS)

$$T^p(\mathcal{F}) = \bigoplus_{i=0}^{\dim \mathbb{P}(V)} E(i-p) \otimes_k H^i(\mathbb{P}(V), \mathcal{F}(p-i))$$

- <u>Ex 1:</u> $N = k$ over E

$$\begin{tikzcd} \hat{E}(-n-1) \otimes_k \Lambda^n V^* \otimes V^* \arrow[r] & \hat{E}(-n) \otimes_k \Lambda^n V^* \arrow[r] \arrow[dr] & \hat{E} \arrow[r] & \hat{E}(-1) \otimes_k V \arrow[r] & \hat{E}(-2) \otimes_k S_2(V) \arrow[r] & \cdots \\ \cdots \arrow[u] & & k \arrow[u] & & & \end{tikzcd}$$

This is a Tate resolution. $(= T(\mathcal{O}_{\mathbb{P}(V)})$, under$)$

- <u>In fact</u>: $D^b(\mathrm{coh}(\mathbb{P}(V))) \xrightarrow{\sim} \mathrm{Tate}(E)$

Ex: $p \in \mathbb{P}^2$, $\mathcal{J}_p \subseteq \mathcal{O}_{\mathbb{P}^2}$.

Cohomology table.

$H^2(\mathbb{P}^2, \mathcal{J}_p(j-2))$	6	3	1	0	0	0	0	0
$H^1(\mathbb{P}^2, \mathcal{J}_p(j-1))$	1	1	1	1	0	0	0	0
$H^0(\mathbb{P}^2, \mathcal{J}_p(j))$	0	0	0	0	2	5	9	14
j	-3	-2	-1	0	1	2	3	4

$$T^{-3}(\mathcal{J}_p) = \hat{E}(5)^6 \oplus \hat{E}(4)^1.$$

- Remark:

$$C(E) \underset{G}{\overset{F}{\rightleftarrows}} C(S)$$

$$\begin{array}{ccc} D^b_{fg}(E) & \underset{G}{\overset{F}{\rightleftarrows}} & D^b_{fg}(S) \\ & & \downarrow \text{ quotient category} \\ \overline{E\text{-mod}} & \simeq & D^b(\mathrm{coh}(\mathbb{P}(V))) \\ \wr\wr & & \\ \mathrm{Tate}(E). & & \end{array}$$