KOSZUL DUALITY SEMINAR
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PARABOLIC – SINGULAR DUALITY:

- **WHAT?** $\mathfrak{g} \supseteq \mathfrak{p} \supseteq \mathfrak{b} \supseteq \mathfrak{h}$

SS/$\mathbb{C}$ — PARABOLIC — BOREL — CARTAN

W – WEYL GROUP
$\cup\!|$
S – SIMPLE REFLECTIONS

- $\mathfrak{p} \rightsquigarrow S_{\mathfrak{p}} \subseteq S$ (SIMPLE REFLECTIONS ASSOC. TO SIMPLE ROOTS IN LEVI $\mathfrak{l} \subseteq \mathfrak{p}$)

- $\lambda \in \mathfrak{h}^* \rightsquigarrow S_\lambda = \{ s \mid s \cdot \lambda = \lambda \} \subseteq S.$

"SPSE $S_\lambda = S_{\mathfrak{p}}$. TO $\lambda, \mathfrak{p}$ WE CAN ASSOCIATE CATEGORIES OF H.W. MODULES $\mathcal{O}_\lambda, \mathcal{O}^{\mathfrak{p}}$. THEN, $\exists$ ISOM. OF $\mathbb{C}$-ALG.

$$\mathrm{End}_{\mathcal{O}_\lambda}(P) \simeq \mathrm{Ext}^{\bullet}_{\mathcal{O}^{\mathfrak{p}}}(L, L)$$

SOME PROJECTIVE GENERATOR — SUM OF ALL SIMPLE OBJECTS

$$\mathrm{End}_{\mathcal{O}^{\mathfrak{p}}}(P) \simeq \mathrm{Ext}^{\bullet}_{\mathcal{O}_\lambda}(L, L)$$

MOREOVER, THESE RINGS ARE KOSZUL DUAL TO EACH OTHER.

$\rightsquigarrow$ REP. THEORY OF $\mathfrak{g}$ GOVERNED BY KOSZUL RINGS

<u>eg</u> WHEN $\mathfrak{p} = \mathfrak{b}$, $S_{\mathfrak{p}} = \emptyset$; CAN TAKE $\lambda = 0$ TO OBTAIN $S_0 = S_{\mathfrak{p}}$ AND $\mathcal{O}_0 = \mathcal{O}^{\mathfrak{b}}$

$$\Rightarrow \mathrm{End}_{\mathcal{O}_0}(P) \simeq \mathrm{Ext}^{\bullet}_{\mathcal{O}_0}(L, L) \text{ IS KOSZUL SELF-DUAL.}$$

"SELF-DUALITY OF TRIVIAL BLOCK"

- **OUTLINE:**
  1) EXAMPLE.
  2) PARABOLIC – SINGULAR DUALITY
  3) KOSZULITY. (SKETCH)

① <u>EXAMPLE</u>: $\mathfrak{g} = \mathfrak{sl}_2 = \mathbb{C}\left\{ \underset{X}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{h}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}, \underset{Y}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} \right\}$

· FIX:

$$\mathfrak{g} \supseteq \mathfrak{b} = \left\{ \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix} \right\} \supseteq \mathfrak{h} = \left\{ \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \right\}, \quad \mathcal{U} = \mathcal{U}(\mathfrak{g})$$

$\underset{\mathbb{C}}{\overset{\lambda \in \mathfrak{h}^*}{\cong}} \rightsquigarrow \underset{\substack{\text{VERMA} \\ \text{W/ H.W } \lambda}}{M(\lambda)},\ \underset{\substack{\text{SIMPLE} \\ \text{W/ H.W } \lambda}}{L(\lambda)}$

- IF $\lambda \in \mathbb{Z}_{<0}$, $M(\lambda) = L(\lambda)$
- IF $\lambda \in \mathbb{Z}_{\geq 0}$, THEN $M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0$

<u>DEFINE</u>: $\mathcal{O}_0$ = FULL SUBCAT. OF $\mathcal{U}$-MOD, W/ OBJECTS M S.T.

1) M F.G.
2) M $\mathfrak{h}$-SEMISIMPLE
3) $\forall m \in M$, $X^n m = 0$, $n >> 0$.
4) $Z \subseteq \mathcal{U}$ ACTS WITH GEN. EVAL 0.

FINITE LENGTH $\mathbb{C}$-CAT. W/ ENOUGH PROJ./INJ.

<u>PROP</u>: (BGG, '76)

- $\lambda \in \{-1, 0, 1, \ldots\}$, $M(\lambda)$ PROJECTIVE IN $\mathcal{O}_0$
- P PROJ., L F.D, $\rightsquigarrow$ $P \otimes L$ PROJ.

· $L \overset{\text{def}}{=} L(0) \oplus L(-2)$; $\left.\begin{array}{l} P(-2) \longrightarrow L(-2) \longrightarrow 0 \\ P(0) \longrightarrow L(0) \longrightarrow 0 \end{array}\right\}$ PROJ. COVERS.

WHERE $P(-2) = M(-1) \otimes L(1)$, $P(0) = M(0)$. $\rightsquigarrow$ $P \overset{\text{def}}{=} P(0) \oplus P(-2)$.

· $\boxed{\mathrm{End}_{\mathcal{O}}(P)}$:

$$\begin{tikzcd} M(0) \arrow[d, "\mathbb{C}"'] \arrow[drr, "\mathbb{C}"] & \oplus & M(-1) \otimes L(1) \arrow[d, "\mathbb{C}"] \\ M(0) & \oplus & M(-1) \otimes L(1) \end{tikzcd}$$

$\Rightarrow \mathrm{End}_{\mathcal{O}}(P) \cong \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\}$

· $\boxed{\mathrm{Ext}^\bullet_{\mathcal{O}}(L, L)}$ USE PROJ. RES.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M(0) & \longrightarrow & P(-2) & \longrightarrow & L(-2) & \longrightarrow & 0 \\ & & \oplus & & \oplus & & \oplus & & \\ 0 & \longrightarrow & M(-1) & \longrightarrow & P(0) & \longrightarrow & L(0) & \longrightarrow & 0 \end{array}$$

③

$$\Rightarrow \quad \mathrm{Ext}^0_{\mathcal{O}}(L,L): \quad \begin{tikzcd} P(-2) \arrow[d, "\mathbb{C}"] & \oplus & P(0) \arrow[d, "\mathbb{C}"] \\ L(-2) & \oplus & L(0) \end{tikzcd} \quad \simeq \mathbb{C}^2.$$

$$\mathrm{Ext}^1_{\mathcal{O}}(L,L) \quad \begin{tikzcd} M(0) \arrow[drr, "\mathbb{C}"'] & \oplus & M(-1) \\ L(-2) & \oplus & L(0) \end{tikzcd} \quad \simeq \mathbb{C}.$$

CHECK: $\mathrm{Ext}^1_{\mathcal{O}}(L,L) \simeq \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\}$.

"EXTENSIONS IN $\mathcal{O}_0$"

• ② PARABOLIC – SING. DUALITY:

SOME DEFINITIONS: $\boxed{\mathfrak{p}_I = \mathfrak{l}_I \oplus \mathfrak{u}_I}$

$$\mathfrak{g} \supseteq \mathfrak{p}_I \supseteq \mathfrak{b} \supseteq \mathfrak{h}$$

ss/$\mathbb{C}$

$$\mathcal{U} = \mathcal{U}(\mathfrak{g}) \supseteq \mathfrak{Z}$$

$W \supseteq S$ ← SIMPLE REFLECTIONS.

$\Phi \supseteq \Phi^+ \supseteq \Delta \supseteq I$.

ROOT DATA (CORRESP. TO $(\mathfrak{b}, \mathfrak{h})$).

$\boxed{\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha}$

eg $\mathfrak{g} = \mathfrak{sl}_n \supseteq \left\{ \begin{bmatrix} \boxed{*} & * \\ 0 & \boxed{*} \end{bmatrix} \right\} = \mathfrak{p}_I \supseteq \mathfrak{b} = \left\{ \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix} \right\} \supseteq \mathfrak{h}$

HERE $I = \Delta - \{\alpha\}$, SOME α.

• DEF'N: $\mathcal{O}_0^{\mathfrak{p}_I}$ = FULL SUBCAT. OF $\mathcal{U}$-MOD w/ OBJECTS M SATISFYING

1) M F.G.
2) $M \underset{\mathcal{U}(\mathfrak{l}_I)}{\simeq} \bigoplus M_r$ ← SIMPLE F.D. $\mathcal{U}(\mathfrak{l}_I)$-MOD.
3) M LOCALLY $\mathfrak{u}_I$-FINITE
4) $\mathfrak{Z}$ ACTS BY TRIVIAL CENTRAL CHARACTER.

• $\exists$ ANALOGS OF VERMAS $M_I(w \cdot 0) \in \mathcal{O}_0^{\mathfrak{p}_I}$, $w \in W$.

AND THEIR SIMPLE QUOTIENTS $L(w \cdot 0) \in \mathcal{O}_0^{\mathfrak{p}_I}$.

MOREOVER:

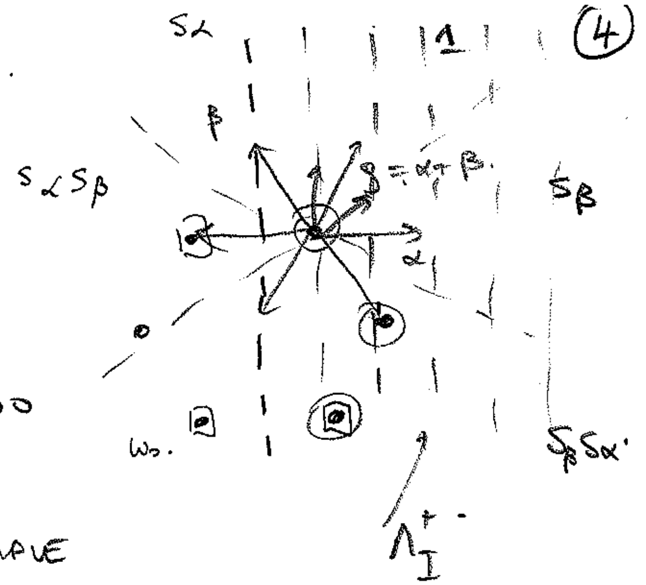
$$\bigoplus_{\alpha \in I} M(s_\alpha w \cdot 0) \longrightarrow M(w \cdot 0) \longrightarrow M_I(w \cdot 0) \longrightarrow 0$$

$M \in \mathcal{O}_0^{\mathfrak{p}_I} \iff$ ALL COMP^N FACTORS $L(\lambda)$ SATISFY $\lambda \in \Lambda_I^+$.

eg $\mathfrak{g} = \mathfrak{sl}_3$, $\Delta = \{\alpha, \beta\} \supseteq I = \{\alpha\}$.

$\cup\!|$

$$\mathfrak{p}_I = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\}.$$



IN THIS CASE:

$$0 \longrightarrow M(s_\alpha w \cdot 0) \longrightarrow M(w \cdot 0) \longrightarrow M_I(w \cdot 0) \longrightarrow 0$$

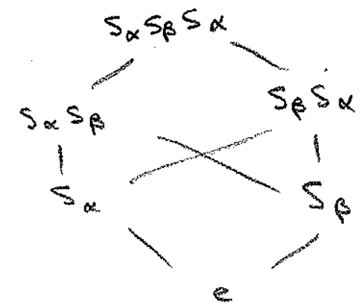
EXACT

$\Rightarrow$

- $M_I(s_\beta s_\alpha \cdot 0) = L(s_\beta s_\alpha \cdot 0)$ SIMPLE
- $M_I(s_\beta \cdot 0)$ HAS COMP^N FACTORS

  $L(s_\beta s_\alpha \cdot 0)$, $L(s_\beta \cdot 0)$.
- $M_I(0)$ HAS COMP'N FACTORS

  $L(0)$, $L(s_\beta \cdot 0)$.



IN GENERAL:

$$\{\text{SIMPLES IN } \mathcal{O}_0^{\mathfrak{p}_I}\} = \{ L(x \cdot 0) \mid x \in W^I \}.$$

{MIN LENGTH REPS OF $W_I \backslash W$}

DENOTE THESE SIMPLES

$$\{ L^I_x, \quad x \in W^I \}$$

& THEIR PROJ. COVERS $P^I_x \in \mathcal{O}^I_x$. (THEY EXIST!)

- NOW, FOR $\lambda \in \mathfrak{h}^*$ WITH $\lambda \in \overline{\Lambda^+} - \rho$, DENOTE.

  ie POSSIBLY SINGULAR

$\mathcal{O}_\lambda$ = FULL SUBCAT. OF $\mathcal{U}$-MOD W/ OBJECTS M SATISFYING

1) M F.G
2) M $\mathfrak{h}$-SEMISIMPLE
3) M LOCALLY $\mathfrak{n}^+$-FINITE
4) $\mathfrak{Z}$ ACTS VIA CENTRAL CHARACTER χ_λ.

DENOTE $S_\lambda = \{ s \in S \mid s \cdot \lambda = \lambda \}$ AND $\{\text{SIMPLES IN } \mathcal{O}_\lambda\} = \{ L(w_0 x^{-1} \cdot \lambda) \mid x \in W^\lambda \}$

W/ PROJ. COVERS $\{ P(w_0 x^{-1} \cdot \lambda), \ x \in W^\lambda \}$.

THEOREM: (BGS) LET $\lambda \in \mathfrak{h}^*$, $I \subseteq \Delta$ BE S.T. $S_\lambda = I$.

THEN, $\exists$ ISOM. OF F.D. $\mathbb{C}$ ALGEBRAS

$$\mathrm{End}_{\mathcal{O}_\lambda}\left(\bigoplus P(w_0 x^{-1} \cdot \lambda)\right) \simeq \mathrm{Ext}^\bullet_{\mathcal{O}^I}\left(\bigoplus L^I_x, \bigoplus L^I_x\right)$$

$$\mathrm{End}_{\mathcal{O}^I}\left(\bigoplus P^I_x\right) \simeq \mathrm{Ext}^\bullet_{\mathcal{O}_\lambda}\left(\bigoplus L(w_0 x^{-1} \cdot 0), \bigoplus L(w_0 x^{-1} \cdot 0)\right)$$

WHERE SUM IS OVER $x \in W^\lambda$

MOREOVER, THESE RINGS ARE KOSZUL DUAL TO EACH OTHER.

IDEA OF PROOF (SEE BGS)

- FOR $\mathfrak{g} \supseteq \mathfrak{p}_I \supseteq \mathfrak{b}$ CONSIDER $G \supseteq P_I \supseteq B$, SEMISIMPLE SIMPLY-CONN.

1) LOCALISATION (BEILINSON-BERNSTEIN):

a) $\exists$ EQUIVALENCE $\mathcal{O}^I \simeq P_B(G/P_I)$

PERVERSE SHEAVES ON G/P_I, W/ COH. LOC. CONSTANT ON B-ORBITS

b) $\exists$ EQUIVALENCE

ALLOWS THE COMPUTATION OF EXTENSIONS IN $D^b(\mathcal{O}^I)$ VIA GEOMETRIC METHODS.

$$D^b(\mathcal{O}^I) \simeq D^b(P_B(G/P_I)) \simeq D^b_B(G/P_I)$$

CXES LOC. CONST. ON B-ORBITS

LAST WEEK.

$$L^I_x \longleftrightarrow IC^I_x.$$

2) GIVE COMBINATORIAL DESCRIPTIONS OF RINGS APPEARING:

a) DIM^{NS} AGREE:

- USE BGG RECIP., KL CONJECTURES, &

$$\dim_{\mathcal{O}^I} \mathrm{Ext}^i(L^I_x, L^I_x) = \dim \mathrm{Hom}_D(IC^I_x, IC^I_x).$$

b) USE SOERGEL BIMODS TO DESCRIBE LHS: $\mathrm{End}_{\mathcal{O}}(\bigoplus P(w_0 x^{-1} \cdot \lambda)) \simeq B_\lambda$

c) USE ABOVE EQUIVALENCE $D^b(\mathcal{O}^I) \simeq D^b_B(G/P_I)$ TO OBTAIN $\mathrm{Ext}^\bullet_{\mathcal{O}^I}(L^I_x, L^I_x) \simeq \mathrm{Ext}^\bullet_D(IC^I_x, IC^I_x)$.

HYPERCOH. GIVES $\mathbb{H}: D(G/P_I) \longrightarrow H^\bullet(G/P_I)$ - grmod. $\stackrel{\text{def}}{=} C^I$

& $\exists$ ISOM. $\mathrm{Ext}^\bullet_{\mathcal{O}^I}(L^I_x, L^I_x) \xrightarrow{\sim} \mathrm{End}_{C^I}(\mathbb{H}^\bullet(\bigoplus IC^I_x))$

d) THE RINGS $\mathrm{End}_{C^I}(\mathbb{H}^\bullet(\oplus IC^I_x))$ AND B_λ ARE ISOM.

WHAT ABOUT KOSZULITY? (SKETCH)

DENOTE $A^I = \mathrm{End}_{\mathcal{O}^I}(\oplus P^I_x)$. So, $\mathcal{O}^I \simeq \text{mod-}A^I$.

- FIRST SHOW $A^\emptyset$ IS SELF-DUAL (i.e. $P_I = b$ CASE)
- IDENTIFY

$$\mathcal{O}^I = \{ M \in \mathcal{O}^\emptyset \mid [M : L^I_x] \neq 0 \Rightarrow x \in W^I \}.$$

$$\rightsquigarrow \mathcal{O}^I \simeq \mathrm{mod}\left(A^\emptyset / J_I\right),$$

WHERE $J_I \subseteq A^\emptyset$ IS TWO-SIDED IDEAL GEN'D BY IDEMPOTENTS $1_x \in A^\emptyset$ WITH $x \in W \setminus W^I$

- CAN IDENTIFY $A^\emptyset / J_I \simeq A^I$; AND USE KOSZULITY OF $A^\emptyset$ TO GET KOSZULITY OF $A^\emptyset / J_I$.
(DETAILS SEE BGS §3.10).