KOSZUL DUALITY SEMINAR 10/10/2013 G. MELVIN.

- AIM OF NEXT 2 TALKS:

EXHIBIT KOSZUL DUALITY PATTERNS BETWEEN VARIOUS CATEGORIES OF REPRESENTATIONS (PARABOLIC - SINGULAR DUALITY)

APPROACH: VIA GEOMETRY OF PARTIAL FLAG VARIETIES. (IE PERVERSE SHEAVES).

- TODAY:

$$\left\{ \begin{array}{l} X = G/B \\ \quad = \coprod_{w \in W} X_w \end{array} \right\} \quad \Big\| \quad D^b(P(X,W)) \underset{\Delta\text{-}d}{\simeq} D^b_c(X,W) \qquad \circledast$$

$P(X,W)$: CORE OF A PERVERSE t-STRUCTURE $({}^pD_c^{\geq 0}, {}^pD_c^{\leq 0})$.

$D^b_c(X,W)$: CONSTRUCTIBLE $\mathbb{C}$-SHEAVES SMOOTH ALONG X_w

OUTLINE:
1. FINITE H.W. CATEGORIES
2. LIGHTNING INTRO TO PERVERSE SHEAVES & IC
3. $\circledast$

---

1. $\mathcal{A}$ - FINITE LENGTH ABELIAN CATEGORY / $\mathbb{C}$

$(S, \leq)$ - FINITE POSET WITH POSET TOPOLOGY.

$T \subseteq S$ CLOSED $\Leftrightarrow \forall s \in T$, $s' \leq s \Rightarrow s' \in T$.

DEF'N: $\mathcal{A}_T$, FULL SUBCAT. OF $\mathcal{A}$ OF OBJECTS SUPP'D ON T.

ASSUME:

(a) $S \overset{\sim}{\longleftrightarrow} \{\text{SIMPLES IN } \mathcal{A}\}$

$s \longleftrightarrow L(s)$

(b) $\mathrm{End}_{\mathcal{A}}(L(s)) = \mathbb{C}$

(c) $\forall s, \exists$ $\Delta(s) \longrightarrow L(s) \longrightarrow 0$

$0 \longrightarrow L(s) \longrightarrow \nabla(s)$ $\quad \in \mathcal{A}$

SUCH THAT:

$\Delta(s) \longrightarrow L(s) \longrightarrow 0$ is PROJ. COVER IN $\mathcal{A}_{\leq s}$

$0 \longrightarrow L(s) \longrightarrow \nabla(s)$ INJ. HULL IN $\mathcal{A}_{\leq s}$

(d) $\ker(\Delta(s) \longrightarrow L(s)) \overset{\text{def}}{=} K_s$
$\in \mathcal{A}_{<s}$.
$\operatorname{coker}(L(s) \longrightarrow \nabla(s)) \overset{\text{def}}{=} C_s$

((e) $\operatorname{Ext}^2(\Delta(s), \nabla(s)) = 0$) NOT NEEDED?

DEF'N – $\Delta(s)$ IS STANDARD OBJECT
– $\nabla(s)$ IS COSTANDARD OBJECT

· PROP'N : • $\| \operatorname{End}_{\mathcal{A}}(\Delta(s)) = \mathbb{C} = \operatorname{End}_{\mathcal{A}}(\nabla(s))$

$(\operatorname{End}_{\mathcal{A}} \Delta(s) = \mathbb{C})$ $\ulcorner$ ① $\operatorname{im} f \subseteq K_s \Rightarrow \operatorname{im} f = 0$ BY (c), (d)
② $\operatorname{im} f \not\subseteq K_s \Rightarrow$ • $\operatorname{im} f|_{K_s} \subseteq K_s$

•
$$\begin{array}{ccc} \Delta(s) & \longrightarrow & L(s) \\ f \downarrow & & \downarrow \bar{f} = c.\mathrm{id}_{L(s)}, \ c \in \mathbb{C} \\ \Delta(s) & \longrightarrow & L(s) \end{array}$$

• $\operatorname{im}(f - c\,\mathrm{id}_{\Delta(s)}) \subseteq K_s$, NOW USE ① $\lrcorner$

• $\| \operatorname{Hom}_{\mathcal{A}}(M,N)$ F.D.

(SKIP?) $\ulcorner$ FIX $M \in \mathcal{A}$; SHOW $\dim \operatorname{Hom}_{\mathcal{A}}(M,N) < \infty$
BY INDUCTION ON $\ell(N)$; REDUCE TO $N = L(s)$; USE INDUCTION ON $\ell(M)$ & (b) $\lrcorner$

• $\| \operatorname{Ext}^1_{\mathcal{A}}(M,N)$ F.D.

$\ulcorner$ REDUCE TO M,N SIMPLE; $M = L(s)$, $N = L(t)$
(SKIP?) WITH $t \not> s \rightsquigarrow \exists\, T \subseteq S$ CLOSED, $s, t \in T$,
s MAXIMAL; APPLY $\operatorname{Hom}(-, L(t))$ TO
$$K_s \hookrightarrow \Delta(s) \twoheadrightarrow L(s).$$
$t > s$ PROCEED DUALLY $\lrcorner$.

THEOREM (A) LET $\mathcal{A}$ BE AS ABOVE. $T \subseteq S$ CLOSED

THEN, $\mathcal{A}_T$ ADMITS ENOUGH PROJECTIVES, AND EACH PROJECTIVE HAS FINITE FILTRATION BY STANDARD OBJECTS. (+ DUAL STATEMENT) (T)

PF: (IDEA)
- INDUCTION ON $|T|$.
- T CLOSED, $s \in S$ S.T. $T' = T \cup \{s\}$ CLOSED.
- REDUCE TO FINDING PROJ. COVER OF $L(t)$, $t \in T'$!
  - $t = s \rightsquigarrow P(s) = \Delta(s)$
  - $t \neq s \rightsquigarrow \exists\, P \in \mathcal{A}_T$ COVERING $L(t)$

SOUGHT FOR $P' \in \mathcal{A}_{T'}$ COMES FROM AN EXTENSION OF $\bigoplus_{i=1}^{r} \Delta(s)$ BY P.

SEE BGS — HOMOLOGICAL ARGUMENTS. ∎

REMARK: IF $(P(s) : \Delta(t)) = \#\ \Delta(t)$ APPEARS IN STANDARD FILTRATION

$\rightsquigarrow$ (RECIPROCITY)

$$(P(s) : \Delta(t)) = [\nabla(t) : L(s)]$$

$$\left(= \dim \operatorname{Hom}_{\mathcal{A}} (P(s), \nabla(t)) \right)$$

SO, $(P(s) : \Delta(t)) \neq 0 \implies t \geq s$.

COROLLARY: $\mathcal{A}$ HAS FINITE COHOMOLOGICAL DIMENSION.

(2) PERVERSE SHEAVES & IC:

MOTIVATION:

(ANDREOTTI - FRANKEL)

V SM. AFFINE / $\mathbb{C}$; $\dim_{\mathbb{C}} V = n$ ($\stackrel{\text{def}}{=}$

$\Rightarrow$ $H^m(V, \underline{\mathbb{Z}}_V) = 0$, $\forall m > n$ $\rightsquigarrow$ (WEAK LEFSCHETZ)

$\Rightarrow$ $H^m_c(V, \underline{\mathbb{Z}}_V) = 0$ $\forall m < n$.

- PERVERSE SHEAF = "CONSTRUCTIBLE SHEAF" F ON V ON V
  S.T.
  $H^m(V, F) = 0$ $m > n$ (ARTIN-GROTH.)
  $H^m_c(V, F) = 0$ $m < n$. (FALSE IN GEN. IF F NOT A CX.)

- FOR REMAINDER OF TALK:

$$X = G/B = \coprod_{w \in W} X_w \qquad T \subseteq B \subseteq G \qquad W = N_G(T)/T.$$

G: s.s. / $\mathbb{C}$

$d_X \stackrel{\text{def}}{=} \dim_{\mathbb{C}} X$

NOTE: • $X_w \subseteq \overline{X_v} \Leftrightarrow w \leq v$ (BRUHAT ORDER.)

eg $G = SL_2$, $X = \mathbb{P}^1_{\mathbb{C}} = \underset{\mathbb{A}^1_{\mathbb{C}}}{\underset{\|}{X_s}} \sqcup \underset{\{*\}}{\underset{\|}{X_e}}$

- DEF'N: • $D^b_C(X, W)$ = FULL SUBCAT. OF $D^b(Sh(X, \mathbb{C}))$ OF OBJECTS $F \in D^b(Sh(X, \mathbb{C}))$ S.T. $H^i(F|_{X_w})$ LOC. CONSTANT (= CONSTANT) ON X_w.

- $D^{\leq 0} = \{ F \in D^b_c(X, W) \mid \dim \operatorname{supp}(H^{-j}(F)) \leq j, \forall j \}$

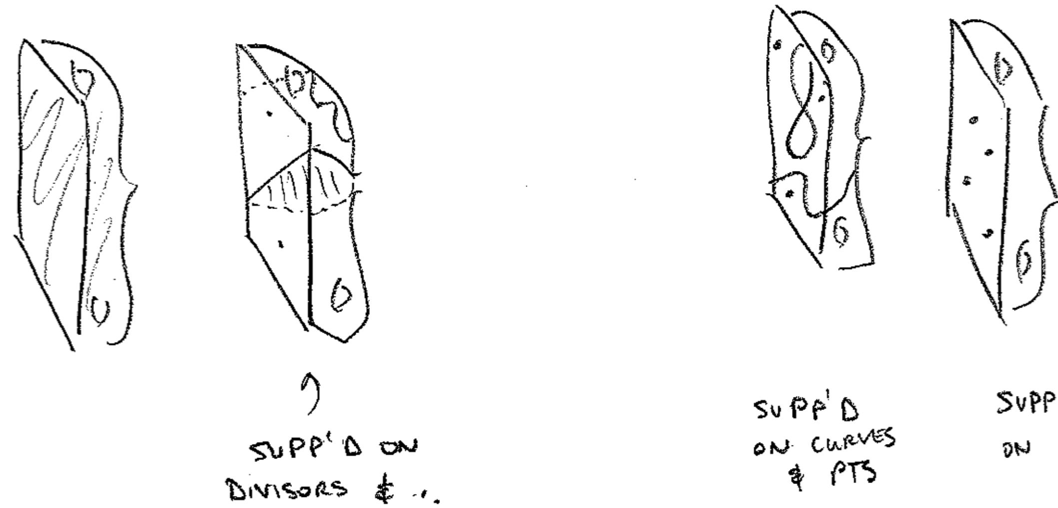
$D^{\geq 0} = \{ F \in D^b_c(X, W) \mid \dim \operatorname{supp}(H^{-j} \mathbb{D} F) \leq j, \forall j \}$.

$\mathbb{D}$: DUALITY FUNCTOR.

ie: $F \in D^{\leq 0} \Rightarrow$

j: $-d_X$ $\quad -d_X+1$ $\quad \ldots$ $\quad -1$ $\quad 0$ $\quad 1$ $\quad 2$

$H^j(F)$



$\emptyset \quad \emptyset$

<u>NOTE</u>: 1) $\mathrm{supp}(H^{-j}DF) = \{x \in X \mid H^{-j}(i_x^! F) \neq 0\}$

$$H^{-j}(i_x^! F) \overset{\text{def}}{=} \varprojlim_{U \ni x} H_c^{-j}(U, F). \quad \text{"COSTALK"}$$

2) $F \in D^{\geq 0} \iff H^j_{X_w}(F) = H^j R\Gamma_{X_w}(F) = 0 \quad \forall j < -d_{X_w}$

• <u>THEOREM</u>: (BBD)

• $(D^{\leq 0}, D^{\geq 0})$ DEFINES t-STRUCTURE ("TRUNCATION") ON $D_c^b(X, W)$.

• $D^{\leq 0} \cap D^{\geq 0} \overset{\text{def}}{=} P(X, W)$ SATISFIES

- ABELIAN
- FINITE LENGTH. $\mathbb{C}$-CATEGORY.

$P(X,W)$ = CAT. OF <u>PERVERSE</u> <u>SHEAVES</u> (SM. ALONG W)

<u>eg</u>: • $\mathbb{C}[d_X] \in P(X, W)$

- $X = SL_2/B = X_s \sqcup X_e$

  $j: X_s \hookrightarrow X$.

CLAIM: $\| \; Rj_*(\mathbb{C}[1]) \in P(X, W)$

$x \in X_s$ $\quad H^m(Rj_*\mathbb{C}[1])_x = H^{m+1}(\mathbb{C})_x = \begin{cases} \mathbb{C} & m=-1 \\ 0 & \text{else} \end{cases}$

$x \in X_e$ $\quad H^m(Rj_*\mathbb{C}[1])_x = H^{m+1}_c(S^1, \mathbb{C}) = \begin{cases} \mathbb{C} & m=-1,0 \\ 0 & \text{else.} \end{cases}$

$\Rightarrow \quad Rj_*\mathbb{C}[1] \in D^{\leq 0}$.

$x \in X_s$ $\quad H_c^m(Rj_*\mathbb{C}[1])_x \cong H^{2-1-m}(X_s, \mathbb{C})_x = \begin{cases} \mathbb{C} & m=1 \\ 0 & \text{else.} \end{cases}$

$x \in X_e$ $\quad H^m_c(Rj_*\mathbb{C}[1])_x \cong H^{2-1-m}(S_1, \mathbb{C})_x = \begin{cases} \mathbb{C} & m=0,1 \\ 0 & \text{else.} \end{cases}$

$\Rightarrow \quad Rj_*\mathbb{C}[1] \in D^{\geq 0}$.

- FACT: $\quad i_w: X_w \longrightarrow X$ GIVES EXACT

$$(i_w)_*: \mathrm{Perv}(X_w, W_{\leq w}) \longrightarrow \mathrm{Perv}(X, W)$$
$$(i_w)_!: \mathrm{Perv}(X_w, W_{\leq w}) \longrightarrow \mathrm{Perv}(X, W)$$

[NEED $X_w \simeq \mathbb{C}^{\ell(w)}$ HERE.

- IC:

(GORESKY-MACPHERSON-DELIGNE; LATE 70s)

- ASSOCIATE TO STRATIFIED PSEUDOMANIFOLD (eg POSSIBLY SINGULAR TRIANGULATED SPACE) THE INTERSECTION HOMOLOGY GPS

$$IH_*(X)$$

(ASSUME MIDDLE PERVERSITY)

- $\exists \; IC_X \in D^b_c(X)$ S.T. $R\Gamma(IC_X)$ COMPUTES IH_*

$$H^i(R\Gamma(IC_X[-d_X])) = IH_i(X)$$

"IC-COMPLEX"

- REMARK: • $IC_x = \mathrm{im}\left((i_{w_0})_!(\mathbb{C}[d_x]) \longrightarrow (i_{w_0})_*(\mathbb{C}[d_x]) \right)$

"MINIMAL EXTENSION"

$$\left.\begin{array}{l} \cdot\ IC_x \in P(X,W) \\ \cdot\ \mathbb{D}(IC_x) = IC_x \\ \cdot\ IC_x\big|_{X_{w_0}} = \mathbb{C}[d_x] \end{array}\right\} \text{CHARACTERISES } IC_x.$$

- LET $IC_w \in P(\overline{X}_w, W_{\leq w})$ AND EXTEND TO X.

$\rightsquigarrow$ $IC_w \in P(X,W)$ BY ABUSE OF NOTATION.

$$\left.\begin{array}{l} \cdot\ IC_w \text{ SELF-DUAL} \\ \cdot\ IC_w\big|_{X_w} = \mathbb{C}[d_{X_w}] \\ \cdot\ \mathrm{supp}\, IC_w = \overline{X}_w \\ \cdot\ \dim \mathrm{supp}\, H^{-j}(IC_w) < m, \quad \forall m < d_{X_w} \end{array}\right\} \text{CHARACTERISES } IC_w.$$

THM: $\{IC_w\}$ ARE SIMPLES IN $P(X,W)$

(3) $D^b(P(X,W)) \simeq D^b(X,W)$ (THM (B))

- USE THEOREM (A) WITH
  - $\mathcal{A} = P(X,W)$
  - $(S, \leq) = (W, \leq)$
  - $L(w) = IC_w$
  - $\Delta(w) = (i_w)_!(\mathbb{C}[d_{X_w}]) \longrightarrow IC_w \longrightarrow 0$
  - $\nabla(w) = (i_w)_*(\mathbb{C}[d_{X_w}]) \longleftarrow IC_w \longleftarrow 0.$

$\Longrightarrow$ $P(X,W)$ HAS ENOUGH PROJECTIVES.

CHECK: $\Delta(w) \to IC_w \to 0$ IS (PROJ.) COVER IN $\mathcal{A}_{\leq w} = P(\overline{X}_w, W_{\leq w})$

FOLLOWS FROM

CLAIM: $\Delta(w)$ HAS NO NONTRIVIAL QUOTIENT SUPP'D ON $X_{<w} = \coprod_{v<w} X_v$.

WHY? IF $K_w * S = \Delta(w)$, WTS $S = \Delta(w)$.

THEN,

$$0 \to S \to \Delta \to J \to 0$$

$\operatorname{Supp} J \subseteq X_{<w}$: ELSE J ADMITS IC_w AS SUBQUOTIENT

$\Rightarrow [\Delta : IC_w] \geq 2$

$\Rightarrow [K_w : IC_w] \geq 1$ $\lightning$.

(AS K_w SUPP'D ON $X_{<w}$).

PF OF THM (B):

BY CONSIDERING $P(X,W)$ AS CAT. OF D-MODS ON X

$$\rightsquigarrow \quad P(X,W) \hookrightarrow D^b(X,W)$$

$$D^b(P(X,W)) \longrightarrow D^b(X,W)$$

BOTH SIDES GEN'D BY PROJ. & INJ. OBJECTS

& $\operatorname{Hom}^i_{D^b(P(X,W))}(P,I) = \operatorname{Hom}^i_D(P,I) \quad \forall i$

(USE P ADMITS STD FILTRATION
I ADMITS COSTD. FILTRATION)