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- Introduction to Schubert polynomials

- Background: - Recall from last time:

- $\text{Gr}^n(\mathbb{C}^m)$  - codimension subspaces of  $\mathbb{C}^m$

- Schubert varieties

$$S_\lambda \subseteq \text{Gr}^n(\mathbb{C}^m)$$

- Schubert classes

$$\sigma_\lambda := [S_\lambda] \in H^{2|\lambda|}(\text{Gr}^n \mathbb{C}^m)$$

and  $\{\sigma_\lambda \mid \lambda \subseteq (m-n) \times n\}$  give basis for  $H^*(\text{Gr}^n \mathbb{C}^m)$ .

- We saw that there is a homomorphism of algebras

$$\textcircled{1} \quad \mathbb{Z}[x, \dots]^{\text{sa}} \xrightarrow{s_\infty} H^*(\text{Gr}^n \mathbb{C}^m)$$

$$S_\lambda \mapsto \begin{cases} \sigma_\lambda & \lambda \subseteq (m-n) \times n \\ 0 & \text{else} \end{cases}$$

"Schur function"

where:  $\textcircled{2} - S_\lambda = \frac{\det(x_i^{\lambda_j + n - j})}{\det(x_i^{n-j})}$

OR  $\textcircled{3} - S_\lambda = \sum_T x^T$   
 $T = \boxed{\text{Young}}$

- Today:  $\text{Gr}^n \mathbb{C}^m \leftarrow F_1(m)$

$$\lambda \subseteq (m-n) \times n \longleftrightarrow w \in S_n$$

$$S_\lambda \longleftrightarrow \tilde{G}_w$$

Outline:

- 1)  $H^*(F_1(m))$
- 2) difference operators
- 3) pipe dreams.

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## Complete flag variety

Define  $F\Gamma(m) = \{ 0 = V_0 \subseteq V_1 \subseteq \dots \subseteq V_m = \mathbb{C}^m \mid \dim V_i = i \}$

Obtained as

$$\text{span}(v_1, \dots, v_i) = (v_i) \iff \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in G\Gamma_m.$$

$$\Rightarrow F\Gamma(m) = B_-^{G\Gamma_m}$$

eg:  $B_- \begin{bmatrix} 4 & 2 & 0 \\ -3 & -1 & 1 \\ 1 & -1 & 2 \end{bmatrix} = B_- \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 3 & 0 & 2 \end{bmatrix}$

$$= B_- \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \in \left\{ \begin{bmatrix} * & 1 & 0 \\ * & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$

- For each permutation  $w \in S_m$ , define  $x_w^\circ = x_{w(1)}^\circ \dots x_{w(m)}^\circ$

$x_w^\circ \subseteq F\Gamma(m)$ ,  $x_w^\circ = \{ V_i \mid \text{RREF corresponds to } w \}$

eg:  $x_{5243}^\circ = \left\{ \begin{bmatrix} * & * & * & * & 1 \\ * & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \right\}$

$$F\Gamma(m) = \coprod_{w \in S_m} x_w^\circ$$

Note: #\*'s = # inversions ( $= l(w)$ )

$$\Rightarrow x_w^\circ \cong \mathbb{C}^{l(w)}$$

The top dimensional cell is  $x_{w_0}^\circ = \left\{ \begin{bmatrix} * & 1 \\ 1 & 0 \end{bmatrix} \right\}$

$$l(w_0) = \binom{m}{2}, \text{ where } w_0 \in S_m \text{ is longest elt.}$$

Define the Schubert variety  $X_w = \overline{X_w^0}$ , and the Schubert class

$$\sigma_w \stackrel{\text{def}}{=} [X_{w,w}] \in H^{2\ell(w)}(Fl(m)). \quad (3)$$

$\{\sigma_w \mid w \in S_n\}$  is a basis for  $H^*(Fl(m))$ .

### Theorem / Definition 1

Let  $S_\infty = \{w: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0} \mid w \text{ has finite support}\}$   
 $= \cup S_m$ .

Then,  $\exists$  a unique family  $\{G_w \mid w \in S_\infty\} \subseteq \mathbb{Z}[x_1, \dots]$   
 such that for  $m \geq 1$ , we have a homomorphism

$$\mathbb{Z}[x_1, \dots, x_m] \longrightarrow H^*(Fl(m)) \quad (+ \text{ some conditions})$$

$$G_w \longmapsto \sigma_w$$

For each fixed  $m \geq 1$ , we have the universal filtration

$$0 = V_0 \subseteq U_1 \subseteq \dots \subseteq U_m = \mathbb{C}^m$$

of the trivial bundle (i.e tautological filtration)

and a homomorphism:

$$\begin{aligned} \mathbb{Z}[x_1, \dots, x_m] &\longrightarrow H^*(Fl(m)) \\ x_i &\longmapsto -c_i(\mathcal{U}_i/\mathcal{U}_{i+1}) \end{aligned}$$

with kernel  $\langle e_n(x) \rangle_{n \geq 0}$ .

## Difference operators:

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- For  $i \geq 1$ , define  $\partial_i : \mathbb{Z}[x_1, \dots, x_n] \hookrightarrow$

$$\partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}$$

eg  $\partial_2(x_1^2 x_2^4 x_3) = x_1^2 x_2 x_3 (x_2^2 + x_2 x_3 + x_3^2)$

- In fact:

- 1)  $G_e = 1$

- 2)  $\partial_i(G_w) = \begin{cases} G_{ws_i} & w(i) > w(i+1) \\ 0 & \text{else.} \end{cases}$

- 3)  $G_{w_0} = x_1^{m-1} \cdots x_{m-1}.$   $x_3 + x_2 + x_1 = 0$

eg:

$$\begin{array}{ccc}
 G_{321} & = x_1^2 x_2 \\
 \downarrow s_1 & & \downarrow s_2 \\
 G_{231} & = x_1 x_2 & G_{312} = x_1^2 \\
 & \downarrow s_2 & \downarrow s_1 \\
 G_{213} = x_1 & & G_{132} = x_1 + x_2 \\
 \downarrow s_1 & & \downarrow s_2 \\
 G_e = 1 & &
 \end{array}$$

Notice: weak Bruhat order appearing

$\{G_w \mid w \in S_m\}$  are a  $\mathbb{Z}$ -basis for  $\mathbb{Z}[x_1, x_2, \dots]$

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Pipe dreams: ("curve diagrams", "graphs")

A pipe dream:  $P$  is a filling of the staircase

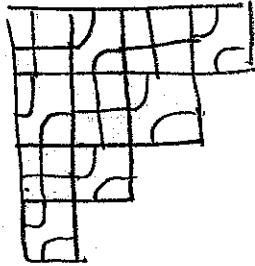


with '+' and ']' (

where

appear on diagonal. Say  $P$  is reduced if no 2 pipes cross twice.

eg:



(not reduced)

To each pipe dream  $P$  of side  $m$  we can associate a permutation  $\text{perm}(P) \in S_m$ .

Theorem / Definition 3 :

for  $w \in S_m$ :

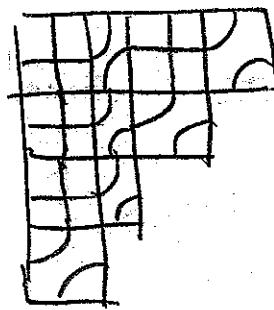
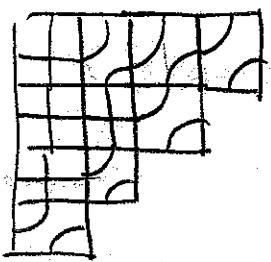
$$G_w = \sum_{\substack{P \text{ reduced, side } m \\ \text{perm}(P) = w}} x^P,$$

where  $x^P = \prod_{i=1}^m x_i^{*\text{ of } '+' \text{ in row } i}$

$\Rightarrow G_w \in \mathbb{N}[x_1, \dots, x_m]$ .

eg: There are 2 pipe dreams corresponding  
to  $2431 \in S_4$

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$$\tilde{G}_{2431} = x_1 x_2^2 x_3 + x_1^2 x_2 x_3$$

- 3 relations with Young tableaux.

- see Knutson: "Schubert polys..."