SOERGEL BIMODULES AND THE SHAPE OF BRUHAT INTERVALS

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ABSTRACT. Given an element w of a Coxeter group, let $a_i(w)$ be the number of elements less than w in Bruhat order. A theorem of Björner and Ekedahl states that if W is crystallographic, then $a_i(w) \leq a_j(w)$ for all $0 \leq i < j \leq \ell(w) - i$. Their proof uses the hard Lefschetz property in intersection cohomology. In this note we extend Björner and Ekedahl's theorem to all Coxeter groups using the hard Lefschetz theorem for Soergel bimodules recently proved by Elias and Williamson. As we explain, the parabolic case remains open.

1. Introduction

Let (W, S) be a Coxeter system with length function ℓ, and let $\leq$ denote the Bruhat order on W. Given $w \in W$, let $[e, w]$ be the interval of elements y between the identity e and w in Bruhat order, and let $a_i(w) := |\{y \in [e, w] \mid \ell(y) = i\}|$ be the number of elements in $[e, w]$ of length i. In this note we prove two theorems, both originally proved by Björner and Ekedahl for crystallographic Coxeter groups. The first concerns the "shape" of the interval $[e, w]$:

Theorem 1.1. *Let $w \in W$. Then*

$$a_i(w) \leq a_j(w) \text{ for all } 0 \leq i \leq j \leq \ell(w) - i. \tag{1}$$

Part of the motivation for studying the shape of $[e, w]$ is a theorem of Carrell and Peterson [Car94], which connects $[e, w]$ to Kazhdan-Lusztig theory. Let $K_{xw}(q)$ denote the Kazhdan-Lusztig polynomial for $x \leq w$. The Carrell-Peterson theorem states that $K_{ew}(q) = 1$ if and only if $[e, w]$ is rank-symmetric, meaning that $a_i(w) = a_{\ell(w)-i}(w)$ for all i. Originally stated, the Carrell-Peterson theorem holds for every Coxeter group satisfying the Kazhdan-Lusztig positivity conjecture, which states that the coefficients of $K_{xw}(q)$ are always non-negative. Recently the Kazhdan-Lusztig positivity conjecture has been proven for all Coxeter groups by Elias and Williamson [EW14], so the Carrell-Peterson theorem holds for all Coxeter groups as well. When W is crystallographic, the elements $w \in W$ index Schubert varieties $X(w)$, and $K_{ew}(q) = 1$ if and only if $X(w)$ is rationally smooth. Equivalently, $X(w)$ is rationally smooth if and only if the intersection cohomology Poincare polynomial $IP_w(q)$ is equal to the Poincare polynomial $P_w(q^2) = \sum_i a_i(w)q^{2i}$ of $[e, w]$.

The second theorem of Björner and Ekedahl relates the failure of rank-symmetry of $[e, w]$ to the first non-zero coefficient of $K_{ew}(q)$.

Theorem 1.2. *Given $w \in W$, let $K_{ew}(q) = 1 + \sum_{k \geq 1} b_k(w)q^k$ and $IP_w(q) = \sum_k c_k(w)q^{2k}$. For any $k_0 \geq 0$ and $\zeta \in \mathbb{Z}$, the following are equivalent:*

(a) $a_k(w) = a_{\ell(w)-k}(w)$ *for all* $0 \leq k < k_0$ *and* $a_{\ell(w)-k_0} - a_{k_0} = \zeta \neq 0$.
(b) $b_k(w) = 0$ *for all* $1 \leq k < k_0$ *and* $b_{k_0}(w) = \zeta \neq 0$.
(c) $a_k(w) = c_k(w)$ *for all* $0 \leq k < k_0$ *and* $c_{k_0}(w) - a_{k_0}(w) = \zeta \neq 0$.

Note that if any of the conditions in Theorem 1.2 holds, then $\zeta > 0$, $a_{\ell(w)-k_0}(w) = c_{\ell(w)-k_0}(w)$, and $a_{k_0}(w) < c_{k_0}(w)$. We defer to Section 4 for the definition of IP_w when W is non-crystallographic.

Björner and Ekedahl's proof of Theorem 1.1 for crystallographic W works as follows: First, they show that the etale cohomology $H^* := H^*(X(w))$ of the Schubert variety $X(w)$ injects into the intersection cohomology $IH^* := IH^*(X(w))$. Furthermore, this injection is equivariant with respect to the action of the cohomology algebra H^* on IH^*. The hard Lefschetz theorem for intersection cohomology states that, if L is multiplication by the first Chern class of an ample line bundle on $X(w)$, then $L^i : IH^{\ell(w)-i} \to IH^{\ell(w)+i}$ is an isomorphism for all $i \geq 0$ (we use the standard cohomological grading, so IH^* is non-zero only in even dimensions and L has degree two). It follows that $L^i : H^{\ell(w)-i} \to H^{\ell(w)+i}$ is injective for all $i \geq 0$. It is well-known that $H^{2k}(X(w))$ has a Schubert basis indexed by the elements $x \in W$ such that $x \leq w$ and $\ell(x) = k$, so Theorem 1.1 follows immediately. The proof of Theorem 1.2 also uses the framework of intersection cohomology.

For non-crystallographic Coxeter groups, Schubert varieties are not defined. To prove Theorems 1.1 and 1.2 for all Coxeter groups, we replace intersection cohomology with Soergel bimodules, and use the hard Lefschetz theorem for Soergel bimodules due to Elias and Williamson [EW14]. Björner and Ekedahl's proofs also apply to the relative Bruhat intervals $[e, w] \cap W^J$, where W^J is the set of minimal length coset representatives of some parabolic subgroup W_J, $J \subset S$. Unfortunately, our proof of Theorems 1.1 and 1.2 for all Coxeter groups does not apply to relative Bruhat intervals, due to the need for a hard Lefschetz theorem for parabolic Soergel bimodules. Whether or not the hard Lefschetz theorem holds for parabolic Soergel bimodules seems to be an interesting open question.

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1.2. **Organization.** In Section 2 we recall the structure ring of a Bruhat interval (analogous to the cohomology ring of a Schubert variety). In Section 3 we explain how this ring is connected with Soergel bimodules and the hard Lefschetz theorem. Theorems 1.1 and 1.2 are proved in Section 4.

2. The structure algebra and Schubert basis

2.1. **Moment graphs and structure algebra.** Let $\mathcal{G}$ be an undirected graph with vertex set $V(\mathcal{G})$ and edge set $E(\mathcal{G})$. A *sheaf* M on $\mathcal{G}$ is a triple

$$\left((M_v)_{v \in V(\mathcal{G})}, (M_e)_{e \in E(\mathcal{G})}, (\rho_{v,e})_{v \in e}\right),$$

where M_a is an abelian group, $a \in V(\mathcal{G}) \cup E(\mathcal{G})$, and $\rho_{v,e} : M_v \to M_e$ is a homomorphism for every vertex v and incident edge e. Given $X \subset V(\mathcal{G})$, the *space of sections* $\Gamma(M, X)$ is the subgroup of $\prod_{x\in X} M_x$ consisting of collections $(f_x)_{x\in X} \in \bigoplus_{x\in X} M_x$ such that $\rho_{x,e}(f_x) = \rho_{y,e}(f_y) \in M_e$ for every edge $e \in E(\mathcal{G})$ joining two vertices $x, y \in X$.

A *moment graph* over a vector space U (which for our purposes we might as well assume to be real) is a pair $(\mathcal{G}, \lambda)$, where $\mathcal{G}$ is an undirected graph together with the additional data of a labelling function $\lambda : E(\mathcal{G}) \to U$. When there is no ambiguity we will denote a moment graph over U by $\mathcal{G}$ (i.e. without reference to λ or U). Let $R = S^*U$ be the symmetric algebra over U. The *structure sheaf* $\mathcal{A}$ *of* $\mathcal{G}$ is the sheaf with $\mathcal{A}_v = R$ for $v \in V(\mathcal{G})$, $\mathcal{A}_e = R/(\lambda(e))$ for $e \in E(\mathcal{G})$, and $\rho_{v,e} : \mathcal{A}_v \to \mathcal{A}_e$ the canonical projection. Note that the structure sheaf only depends on the lines spanned by the $\lambda(e)$'s, not on the actual vectors. Since R is a ring, $\Gamma(\mathcal{A}, X)$ is a ring under pointwise multiplication for any $X \subset V(\mathcal{G})$. In particular,

$$\mathcal{R} := \Gamma(\mathcal{A}, V(\mathcal{G})) = \left\{(f_x)_{x\in V(\mathcal{G})} \subset R^{V(\mathcal{G})} \ : \ \lambda(xy) \text{ divides } f_x - f_y \text{ for all } xy \in E(\mathcal{G})\right\}$$

is a ring, called the *structure algebra of* $(\mathcal{G}, \lambda)$. The ring $\mathcal{R}$ is an algebra over R via the diagonal embedding $\Delta : R \to \mathcal{R}$. For more background on moment graphs, we refer to the surveys [Jan10] and [Fie13].

2.2. **The Bruhat graph.** Let (W, S) be a Coxeter system with Coxeter matrix $(m_{s,t})_{s,t\in S}$, and let $\mathcal{T} := \bigcup_{x\in W} xSx^{-1}$ be the set of reflections of W. Take a finite dimensional real vector space V with linearly independent subsets $\{\alpha_s\}_{s\in S} \subset V^*$ and $\{\alpha_s^\vee\}_{s\in S} \subset V$ such that

$$\langle \alpha_s, \alpha_t^\vee \rangle = -2\cos(\pi/m_{s,t}) \quad \text{ for every } s, t \in S,$$

and such that $\dim V$ is minimal with respect to this condition. Then W acts on V by $s \cdot v = v - \alpha_s(v)\alpha_s^\vee$, and $V_0 = \operatorname{span}\{\alpha_s^\vee : s \in S\} \subset V$ is the geometric representation of W. Similarly, if $f \in V^*$, then $s \cdot f = f - f(\alpha_s^\vee)\alpha_s$, and $\operatorname{span}\{\alpha_s : s \in S\}$ is also isomorphic to V_0. We work with V and V^*, rather than the geometric representation V_0, because in the next section we will need the fact that V is a reflection-faithful representation in the sense of Soergel [Soe07]. Because V_0 is the standard representation, $\Phi = W \cdot \{\alpha_s : s \in S\} \subset V_0$ is a root system for W, and Φ can be partitioned into positive and negative roots Φ^+ and Φ^- respectively. In addition, there is a bijection between reflections $\mathcal{T}$ and positive roots Φ^+ sending $t = wsw^{-1}$ to $\alpha_t := w\alpha_s$.

The *Bruhat graph* $\mathcal{G}_W$ of (W, S) is the graph with vertex set W, and an edge between $x, y \in W$ if and only if $tx = y$ for some $t \in \mathcal{T}$. Because each edge xy is labelled by a unique reflection $t \in \mathcal{T}$, the Bruhat graph can be regarded as a moment graph (with infinitely many vertices) over V^* via the labelling function $\lambda(xy) = \alpha_t$, where $tx = y$. We use $\mathcal{R}_W$ to denote the structure algebra of $\mathcal{G}_W$. The Bruhat graph of an interval $[e, w]$ is the subgraph $\mathcal{G}_w$ of $\mathcal{G}_W$ induced by the vertices $[e, w]$. We let $\mathcal{R}_w$ denote the structure algebra of $\mathcal{G}_w$, where $\mathcal{G}_w$ is regarded as a moment graph via the restriction of λ. In contrast to $\mathcal{G}_W$, the graph $\mathcal{G}_w$ is always finite. We can now state the main structure theorem for $\mathcal{R}_w$:

Theorem 2.1. *Given $w, x \in W$, let $\underline{x} = s_1 \dots s_n$ be a reduced expression, where $s_i \in S$. Define*

$$\xi_w(\underline{x}) = \begin{cases} \sum_{\substack{1 \le i_1 < \dots < i_{\ell(w)} \le n \\ s.t.\ s_{i_1} \cdots s_{i_{\ell(w)}} = w}} \beta_{i_1} \cdots \beta_{i_{\ell(w)}} & w \le x \\ 0 & w \not\le x \end{cases},$$

where $\beta_j = s_1 \cdots s_{j-1}(\alpha_{s_j})$, $j \ge 1$. Then:

(a) $\xi_w(\underline{x})$ is independent of the choice of reduced expression for x, giving rise to a well-defined function

$$\xi_w : W \to R,$$

(b) if $w \le x$ then $\deg \xi_w(x) = \ell(w)$, so in particular $\xi_w(x) \neq 0$, and
(c) the set of functions $\{\xi_y,\ y \le w\}$, when restricted to $[e, w]$, form an R-module basis for $\mathcal{R}_w$. Thus, $\mathcal{R}_w$ is a graded free R-module.

The basis $\{\xi_w : w \in W\}$ is called the *Schubert basis* of $\mathcal{R}_w$. An example calculation of the Schubert basis is given at the end of the section. When W is crystallographic, Theorem 2.1 is due to Kostant-Kumar [KK86] and Billey [Bil99]. The formula for $\xi_w(x)$ in part (a) is often called *Billey's formula* [Tym13]. The proof of Theorem 2.1 in [KK86] and [Bil99] can be readily extended to all Coxeter groups. For the convenience of the reader, we give a streamlined proof of Theorem 2.1, based on [KK86, Bil99] and an unpublished paper of Stembridge [Ste93]. We start with the proof of part (a).

Let $I = \{s, t\} \subset S$ be such that $m := m_{s,t} < \infty$. Then, $\operatorname{span}\{\alpha_s, \alpha_t\} \subset V^*$ gives the geometric representation of the subgroup $W_I \subset W$ generated by $\{s, t\}$. There are exactly two reduced expressions for the longest element $w_0 \in W_I$, $w_0 = st \dots = ts \dots$, each having length m. Fix the reduced expression $\underline{w}_0 = st \dots$, and let $\beta_j \in \operatorname{span}\{\alpha_s, \alpha_t\}$ be defined as in the statement of Theorem 2.1. Thus, $(\beta_1, \dots, \beta_m)$ is an ordering of the positive roots in $\operatorname{span}\{\alpha_s, \alpha_t\}$. The root sequence for the other reduced expression is $(\beta_m, \dots, \beta_1)$.

Define N to be the associative $\mathbb{R}$-algebra generated by $\{a_s, a_t\}$, such that $a_s^2 = a_t^2 = 0$, and with the following *braid relation* $a_s a_t a_s \dots = a_t a_s a_t \dots$ (m-fold product); N is the *nil Coxeter algebra* of W_I. For any $w = s_{i_1} \cdots s_{i_l} \in W_I$, the braid relation ensures that the element $a_w = a_{i_1} \cdots a_{i_l} \in N$ is well-defined. Consider the polynomials $h(z) = (1 + z \otimes a_s)$, $g(z) = (1 + z \otimes a_t) \in \mathbb{R}[z] \otimes N$. Then, for any $w \in W_I$, it follows that the formula for $\xi_w(\underline{w}_0)$ given in Theorem 2.1 is the coefficient of a_w in the (m-fold) product $h(\beta_1)g(\beta_2)h(\beta_3) \cdots \in R \otimes N$.

Lemma 2.2 (Theorem 3, [Ste93])**.** *Let $h(z), g(z) \in \mathbb{R}[z] \otimes N$, $\beta_1, \dots, \beta_m \in \Phi^+$ be the elements from the previous paragraph. Then, $h(\beta_1)g(\beta_2) \cdots = g(\beta_m)h(\beta_{m-1}) \cdots \in R \otimes N$. In particular, $\xi_w(\underline{w}_0)$ is independent of the choice of reduced expression for $w_0 \in W_I$.*

Lemma 2.3. *The functions $\xi_w(\underline{x})$ are independent of the choice of reduced expression for x.*

Proof. Suppose that x has a reduced factorization $x = x_1x_2x_3$, where $\underline{x}_2 = st\cdots$ is an s,t braid of length $m_{s,t}$. Then

$$\xi_w(\underline{x}) = \sum_{\substack{y_i \le x_i \\ \text{s.t. } w=y_1y_2y_3, \\ \ell(w)=\ell(y_1)+\ell(y_2)+\ell(y_3)}} \xi_{y_i}(\underline{x}_i)\left(x_1 \cdot \xi_{y_2}(\underline{x}_2)\right)\left(x_1x_2 \cdot \xi_{y_3}(\underline{x}_3)\right).$$

It is well known that all reduced expressions for x can be obtained from $\underline{x}$ via braid moves $sts\cdots \to tst\cdots$ so that the claim will follow by induction, provided it holds for (W,S) with $|S| = 2$, and $x = st\cdots = ts\cdots$. This follows from Lemma 2.2. □

Now that we know that $\xi_w : W \to R$ is a well-defined function, let Q be the fraction field of R, and extend the action of W on R to Q. The *twisted group ring* Q_W is the Q-vector space $\oplus_{w\in W} Qw$, with multiplication defined by

$$qwqw' = qw(q') \cdot ww' \in Qww'$$

Let $\mathcal{F}$ denote the Q-vector space of functions $f : W \to Q$. Then there is an action of Q_W on $\mathcal{F}$ given by

$$((qw)f)\,(u) = qw(f(w^{-1}u)),$$

and this allows us to define the *Demazure operator* $D_s := \frac{1}{\alpha_s}(1-s) \in Q_W$ for any $s \in S$. These operators satisfy $sD_s = D_s = -D_ss$, so in particular $D_s^2 = 0$.

Note that the elements ξ_w belong to $\mathcal{F}$. Given $w \in W$, let $\mathrm{D}_L(w)$ denote the left descent set of w.

Proposition 2.4. *For any $s \in S, w \in W$, we have*

$$D_s\xi_w = \begin{cases} \xi_{sw} & s \in \mathrm{D}_L(w) \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. Suppose that $s \in \mathrm{D}_L(w)$ and let $v, u \in W$ be such that $v = su$, $\ell(v) = \ell(u)+1$. Then,

$$\begin{aligned}(D_s\xi_w)(v) &= \frac{1}{\alpha_s}\left(\xi_w(v) - s\xi_w(u)\right) \\ &= \frac{1}{\alpha_s}\left(\alpha_s\xi_{sw}(v)\right) \\ &= \xi_{sw}(v),\end{aligned}$$

while

$$\begin{aligned}(D_s\xi_w)(u) &= \frac{1}{\alpha_s}\left(\xi_w(u) - s\xi_w(v)\right) \\ &= \frac{1}{\alpha_s}\left(-s\left(\alpha_s\xi_{sw}(v)\right)\right) \\ &= \xi_{sw}(u).\end{aligned}$$

Hence $D_s\xi_w = \xi_{sw}$ for any $s \in \mathrm{D}_L(w)$. If $s \notin \mathrm{D}_L(w)$, then we have $\xi_w = D_s\xi_{sw}$ and $D_s\xi_w = D_s^2\xi_{sw} = 0$. □

Proof of Theorem 2.1. We have already shown part (a) in Lemma 2.3. Part (b) follows from the definition and the fact that β_j is a positive root, and hence is a non-negative linear combination of α_s's.

For part (c), we first have to show that ξ_y belongs to R_w for all $y \leq w$. By Proposition 2.4 we see that $D_{s_1} \cdots D_{s_n} \xi_y$ is R-valued, for any sequence $s_1, \ldots, s_n \in S$. Moreover, it is clear that

$$Q_W = \operatorname{span}_Q \{D_{s_1} \cdots D_{s_n} \mid s_1, \ldots, s_n \in S\}$$

and that

$$N := \operatorname{span}_R \{D_{s_1} \cdots D_{s_n} \mid s_1, \ldots, s_n \in S\}$$

contains $W \subset Q_W$. Thus $n\xi_y$ is R-valued for any $n \in N$.

Let $\alpha_t \in \Phi$, so that $\alpha_t = u(\alpha_s)$ for some $u \in W, s \in S$. Then

$$uD_s u^{-1} = u \left(\frac{1}{\alpha_s}(1-s) \right) u^{-1} = \frac{1}{\alpha_t}(1-t),$$

and $uD_s u^{-1} \in N$ implies $\frac{1}{\alpha_t}(1-t)\xi_w$ is R-valued. If $v = tu$ for some $t \in T$ then

$$\frac{1}{\alpha_t} \left(\xi_w(v) - t\xi_w(u) \right) \in R,$$

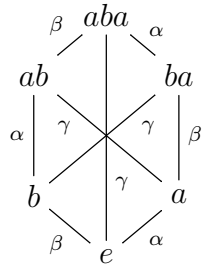
and this happens if and only if

$$\alpha_t \text{ divides } \left(\xi_w(v) - \xi_w(u) + \xi_w(u) - t\xi_w(u) \right).$$

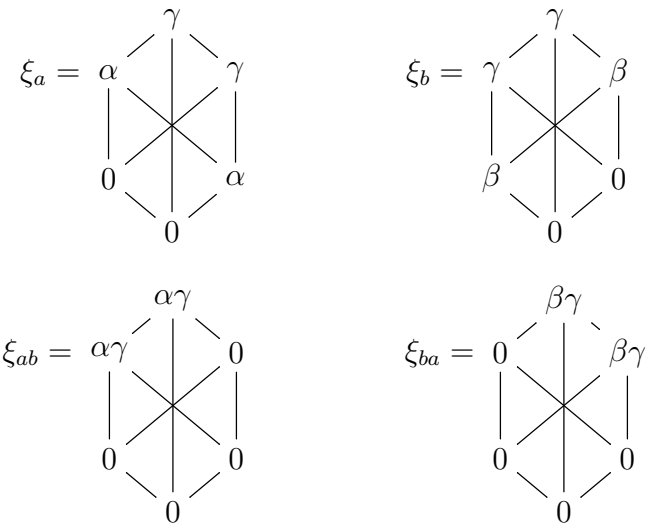
Since α_t divides $\xi_w(u) - t\xi_w(u)$, we conclude that $\xi_w \in \mathcal{R}$ as desired.

It remains to show that $\{\xi_y\}_{y \leq w}$ is an R-basis of $\mathcal{R}_w$. For this, choose a linear extension $y_1 \prec \ldots \prec y_r$ of the interval $[e, w]$. Define the *support of* $f = (f_y)_{y \leq w}$ to be $\operatorname{supp}(f) = \{y \leq w \mid f_y \neq 0\}$. Then $\{\xi_y\}_{y \leq w}$ is linearly independent over R by support considerations. Moreover, if $f = (f_y)_{y \leq w} \in \mathcal{R}_w$ and $\operatorname{supp}(f) \subset \{y \leq y_i\}$ with $f_{y_i} \neq 0$, then $f_{y_i} = p\alpha_{j_1} \cdots \alpha_{j_r}$, for some $p \in R$, where α_{j_k} are the labels attached to any edge e with endpoints y_i and x_k, with $\ell(x_k) < \ell(y_i)$. Thus, $\operatorname{supp}(f - p\xi_{y_i}) \subset \{y < y_i\}$ and $f \in \sum_j R\xi_{y_j}$ by induction on i. □

Example 2.5. *Let $W = S_3, S = \{a, b\}$ with positive roots $\{\alpha, \beta, \alpha + \beta\}$, so that a is the reflection corresponding to the root α, b is the reflection corresponding to the root β. Set $\gamma = \alpha + \beta$. The moment graph is*



We have the following elements of the Schubert basis



3. Braden-MacPherson sheaves and Hodge theory

3.1. **Braden-Macpherson sheaves.** Let $(\mathcal{G}, \lambda)$ be a *directed* moment graph. For $\mathcal{G}$ acyclic we obtain a partial ordering $\leq$ on $V(\mathcal{G})$: $x \leq y \in V(\mathcal{G})$ if and only if there is a directed edge $x \xrightarrow{e} y \in E(\mathcal{G})$. For any $x \in V(\mathcal{G})$ we write $\{> x\} = \{y \in V(\mathcal{G}) \mid y > x\}$, and similarly for $\{\geq x\}$, $\{< x\}$, $\{\leq x\}$.

Suppose that $\mathcal{G}$ is a finite, acyclic directed graph and $V(\mathcal{G})$ has a unique highest element w_0 with respect to the induced partial order.

The *Braden-Macpherson sheaf on* $\mathcal{G}$ (or BM-sheaf when there is no confusion) is the sheaf $\mathcal{BM}(\mathcal{G})$ on $\mathcal{G}$ constructed inductively as follows:

(1) Set $\mathcal{BM}(\mathcal{G})^{w_0} = R$.
(2) Suppose $\mathcal{BM}(\mathcal{G})^y$ has been constructed already, for some $y \in V(\mathcal{G})$. Define
$$\mathcal{BM}(\mathcal{G})^e := \mathcal{BM}(\mathcal{G})^y / \lambda(e)\, \mathcal{BM}(\mathcal{G})^y$$
for any directed edge e ending at y, and let $\rho_{y,e}$ be the canonical quotient homomorphism.
(3) Suppose $\mathcal{BM}(\mathcal{G})$ has been constructed on the full subgraph $\{> x\}$. Define
$$\mathcal{BM}(\mathcal{G})^{\delta x} := \operatorname{im}\left(\Gamma(\mathcal{BM}(\mathcal{G}), \{> x\}) \to \oplus_{e \in E(\mathcal{G})^{\delta x}} \mathcal{BM}(\mathcal{G})^e\right)$$
Here $E(\mathcal{G})^{\delta x} = \{e \in E(\mathcal{G}) \mid x \xrightarrow{e} y \text{ for some } y \in V(\mathcal{G})\}$, the set of edges starting at x, and the map is the canonical projection. Define $\mathcal{BM}(\mathcal{G})^x$ to be a (graded) projective (i.e. free) cover of $\mathcal{BM}(\mathcal{G})^{\delta x}$, and $\rho_{x,e}$ are the components of the map $\mathcal{BM}(\mathcal{G})^x \to \mathcal{BM}(\mathcal{G})^{\delta x} \subset \oplus_{e \in E(\mathcal{G})^{\delta x}} \mathcal{BM}(\mathcal{G})^e$.

The global sections $BM(\mathcal{G}) := \Gamma(\mathcal{BM}(\mathcal{G}), V(\mathcal{G}))$ of $\mathcal{BM}(\mathcal{G})$ will be called the *Braden-Macpherson module on* $\mathcal{G}$, or *BM-module* when there is no confusion. By construction, $BM(\mathcal{G})$ is a module over the stucture algebra $\mathcal{R}$ of $\mathcal{G}$. In particular, $BM(\mathcal{G})$ admits an R-module structure coming from the diagonal embedding $\Delta : R \to \mathcal{R}$. We will call this the *standard* R-module structure.

When $\mathcal{G}_w$ is the moment graph of the Bruhat interval $[e, w]$ we write $\mathcal{BM}(w)$ for the BM-sheaf. For any $x \in W$, $\mathcal{BM}(w)^x$, contains a unique degree 0 summand. Hence, the corresponding BM-module, $BM(w)$, contains a (unique) free $\mathcal{R}_w$-submodule. Moreover, $BM(w)$ is (graded) free with respect to the standard R-module structure.

If $\mathcal{G}$ is the moment graph of 0 and 1-dimensional orbits of a sufficiently nice irreducible, complex T-variety X (T an algebraic torus), Braden-Macpherson showed in [BM01] that $BM(\mathcal{G})$ is isomorphic to the T-equivariant intersection cohomology $IH_T^*(X)$ as a graded $H^*(BT)$-module. This isomorphism induces $H_T^*(X) \cong \mathcal{R}$. The non-equivariant intersection cohomology is obtained as $IH^*(X) \cong BM(\mathcal{G})/U \cdot BM(\mathcal{G})$, with $H^*(X) \cong \mathcal{R}/U \cdot \mathcal{R}$ and intertwine the action of $H^*(X)$ on $IH^*(X)$ with the action of $\mathcal{R}$ on $BM(\mathcal{G})$.

3.2. **Soergel bimodules.** Let V be a reflection faithful representation of the Coxeter system (W, S), R the ring of regular functions on V. Consider the $\mathbb{Z}$-grading on R obtained by letting V^* sit in degree 2. We work in R-$_{\mathbb{Z}}$mod-R, the category of R-bimodules that are $\mathbb{Z}$-graded as left R-modules.

In [Soe07], Soergel introduces a category of R-bimodules $\mathcal{B} \subset R$-$_{\mathbb{Z}}$mod-R. He shows that $\mathcal{B}$ provides a manifestation of the Hecke category of (W, S): it is an additive monoidal category with split Grothendieck group isomorphic to the Hecke algebra $\mathcal{H}$ of (W, S). We denote the resulting isomorphism ch : $K(\mathcal{B}) \to \mathcal{H}$, the *character map.*

Soergel constructs certain 'special' bimodules $B(x) \in \mathcal{B}$, $x \in W$, and conjectures that the image of their isomorphism classes under the character map should coincide with the Kazhdan-Lusztig basis. This conjecture was verified by Elias-Williamson in [EW14].

Conjecture 3.1 (Soergel conjecture)**.** [Soe07][EW14] *Let (W, S) be a Coxeter system, $\mathcal{H}$ the associated Hecke algebra. Then, $\{\mathrm{ch}(B(x)) \mid x \in W\} \subset \mathcal{H}$ recovers the Kazhdan-Lustig basis.*

3.3. **The isomorphism between BM-modules and Soergel bimodules.** Fiebig gave a characterisation of $\mathcal{B}$ in the language of $\mathcal{R}_W$-modules and sheaves on the Bruhat graph [Fie08].

Write xf for the standard W action on R and Q ($x \in W$, $f \in Q$). Then, the structure algebra $\mathcal{R}_W$ ($w \in W$) admits a *twisted* R-module structure: for $f \in R$, we let $\sigma(f) = (xf)_{x \in W}$. Then, $\sigma(f) \in \mathcal{R}_W$ and we get a well-defined homomorphism $\sigma : R \to \mathcal{R}_W$. In this way, $BM(w)$ becomes an R-bimodule via restriction $\Delta \otimes \sigma : R \otimes R \to \mathcal{R}_W$.

For a graded R-module $M = \bigoplus_i M^i$, we denote $M\{k\} = \bigoplus_i M\{k\}^i$, $k \in \mathbb{Z}$, to be the graded R-module with $M\{k\}^i = M^{i+k}$.

Proposition 3.2. [Fie08] *Let $w \in W$, Then, there is an isomorphism of R-bimodules $BM(w) \cong B(w)\{-\ell(w)\}$.*

Fiebig gives an equivalent reformulation of Soergel's conjecture (Conjecture 3.1) in the language of moment graphs.

Conjecture 3.3 (Soergel conjecture for moment graphs)**.** *If* $\mathcal{BM}(w)^x \cong \bigoplus_i R\{k_i\}$ *then* $K_{x,w}(v^2) = \sum_i v^{-k_i}$*, where* $K_{x,w}$ *are the* Kazhdan-Lusztig polynomials*.*

3.4. **The work of Elias-Williamson.** Elias-Williamson provide a proof of Conjecture 3.1 (equivalently Conjecture 3.3), identifying $\{\mathrm{ch}(B(x)) \mid x \in W\}$ with the Kazhdan-Lusztig basis of $\mathcal{H}$.

An essential part of Elias-Williamson's work is obtaining an analog of the hard Lefschetz theorem for Soergel bimodules.

Theorem 3.4. [EW14] *Let* $\rho \in V^*$ *satisfy* $\langle \rho, \alpha_s^\vee \rangle > 0$*, for each* $s \in S$*. Denote by* $\overline{\rho}$ *the* $\mathbb{R}$*-linear operator on* $\overline{B(x)} := \mathbb{R} \otimes_R B(x)$ *induced by the right action of* ρ *on* $B(x)$*. Then,*

$$\overline{\rho}^i : \left(\overline{B(x)}\right)^{-i} \to \left(\overline{B(x)}\right)^{i}$$

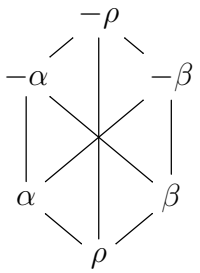
is an isomorphism for each $i \geq 0$*.*

As a consequence, they obtain an extension to arbitrary Coxeter systems of the following:

Corollary 3.5 (Kazhdan-Lusztig positivity conjecture)**.** [**?**] *Let* (W, S) *be a Coxeter system,* $K_{x,w} \in \mathbb{Z}[v^{\pm 1}]$ *a Kazhdan-Lusztig polynomial. Then, the coefficients of* $K_{x,w}$ *are nonnegative integers.*

Example 3.6. *Let* $\rho = \alpha + \beta$*. Then,*

$\sigma(\rho) =$



Thus, $(1 \otimes \rho)\xi_a = \sigma(\rho)\xi_a = \beta\xi_a - 2\xi_{ba} - \xi_{ab}$*. Similarly,* $(1 \otimes \rho)\xi_b = \alpha\xi_b - \xi_{ba} - 2\xi_{ab}$*. The matrix of* $\overline{\rho}$ *in the Schubert basis is* $\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$.

4. Proof of main theorems

4.1. **Proof of Theorem 1.1.** Let $w \in W$ and $BM(w) = \oplus_i BM(w)^i$ the corresponding BM-module, where $BM(w)^i$ is the degree i summand of the (free) left R-module $BM(w)$. Since $\mathcal{R}_w$ is an $\mathcal{R}_w$-submodule of $BM(w)$, Theorem 2.1 provides a linearly independent subset $\{\xi_x\}_{x \leq w}$ in $BM(w)$, with respect to the left R-module structure. Moreover, $\xi_x \in BM(w)^{\ell(x)}$, for each $x \leq w$. Hence,

$$\dim_{\mathbb{R}} \left(\overline{BM(w)}\right)^i \geq a_i(w), \text{ for } i \geq 0.$$

Let ρ be as in Theorem 3.4. Then, $\sigma(\rho) \in \mathcal{R}_w$ so that the $\mathcal{R}_w$-submodule $\mathcal{R}_w \subset BM(w)$ is preserved by multiplication by $\sigma(\rho)$. The result follows from Propositition 3.2 and Theorem 3.4.

4.2. **Proof of Theorem 1.2.** For a (graded) free R-module $M \cong S\{k_i\}$ we denote the *graded rank* of M by $\underline{\text{rank}}\ M = \sum_i v^{-k_i} \in \mathbb{Z}[v^{\pm 1}]$. For $x \leq w \in W$, define

$$IP_w := \underline{\text{rank}}\ BM(w) = \sum c_i(w) v^{2i}.$$

Observe that $P_w = \sum_{x \leq w} v^{2\ell(x)} = \sum_i a_i(w) v^{2i}$, by Theorem 2.1, and $IP_w(v) = v^{2\ell(w)} IP_w(v^{-1})$, by Theorem 3.4 and Proposition 3.2.

It is a consequence of Soergel's conjecture (Conjecture 3.3) that $\underline{\text{rank}}\ \mathcal{BM}(w)^x = K_{x,w}(v^2)$, where $K_{x,w} \in \mathbb{Z}[v^{\pm 1}]$ is the Kazhdan-Lusztig polynomial. We write

$$K_{e,w}(v^2) = \sum_i b_i(w) v^{2i} = 1 + b_1(w) v^2 + \ldots + b_r(w) v^{2r}.$$

Corollary 3.5 shows that $b_1(w), \ldots, b_r(w) \in \mathbb{Z}_{\geq 0}$.

Proposition 4.1. *Let $w \in W$.*

(i) $IP_w(v) = \sum_{x \leq w} v^{2\ell(x)} K_{x,w}(v^2)$,
(ii) (Monotonicity) *If $y \leq x$ then $K_{y,w}(v) - K_{x,w}(v) \in \mathbb{Z}_{\geq 0}[v]$,*
(iii) The following are equivalent:
  (a) $b_{k_0}(w) \neq 0$ and $b_k(w) = 0$, for $0 < k < k_0$,
  (b) for $k < k_0$, $a_k(w) = c_k(w)$ and $a_{k_0}(w) < c_{k_0}(w)$,
  (c) for $k < k_0$, $a_k(w) = a_{\ell(w)-k}(w)$, and $a_{\ell(w)-k_0}(w) - a_{k_0}(w) = b_{k_0}(w) > 0$.

Proof. (i) Let $\{\leq w\} = \{x_1, \ldots, x_n\}$ so that $x_i < x_j$ (in Bruhat order) implies $i < j$. Denote $\Omega_j = \{x_i \mid i \geq j\}$ and define $F_j := BM(w)^{\Omega_j}$. Observe that each Ω_j is upwardly closed. We obtain a cofiltration (F_i) of $BM(w)$

$$BM(w) = F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} 0$$

Furthermore, if $\mathcal{BM}(w)^{x_i} \cong \bigoplus_{j=1}^{m_i} S\{k_{i,j}\}$ then $K_i := \ker f_i \cong \bigoplus_{j=1}^{m_i} S\{2(\ell(x_i) - \ell(w)) - k_{i,j}\}$ [Fie08]. Therefore,

$$IP_w = \sum_{i=1}^{n} \underline{\text{rank}}\ K_i = \sum_{i=1}^{n} \sum_{j=1}^{m_i} v^{k_{i,j} - 2(\ell(x_i) - \ell(w))}$$

and, using $IP_w(v) = v^{2\ell(w)} IP_w(v^{-1})$, we find

$$IP_w(v) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} v^{2\ell(x_i) - k_{i,j}} = \sum_{x \leq w} v^{2\ell(x)} K_{x,w}(v^2).$$

(ii) Let $x \leq y \leq w$, with $\ell(x) + 1 = \ell(y)$, and denote the edge $E : x \to y$. The construction of $\mathcal{BM}(w)$ implies that there is a surjective R-module homomorphism $\mathcal{BM}(w)^x \to \mathcal{BM}(w)^{\delta x} \to \mathcal{BM}(w)^E$, where the last map is projection onto the E summand. Upon tensoring with the trivial (graded)

R-module $\mathbb{R}\{0\}$, the quotient homomorphism $\mathcal{BM}(w)^y \to \mathcal{BM}(w)^E$ becomes an isomorphism of graded vector spaces

$$\mathcal{BM}(w)^y \otimes_R \mathbb{R} \cong \mathcal{BM}(w)^E \otimes_R \mathbb{R} \cong \bigoplus_j \mathbb{R}\{k_j\}.$$

The existence of the surjection $\mathcal{BM}(w)^x \otimes_R \mathbb{R} \to \mathcal{BM}(w)^E \otimes_R \mathbb{R}$ gives the result.

(iii) Let $q = v^{1/2}$. We will consider $P_w(q), IP_w(q), K_{e,w}(q)$ for ease of notation. Let $f[j] \in \mathbb{Z}$ denotes the coefficient of q^j in $f \in \mathbb{Z}[q]$. Then, (i) implies that

$$c_j(w) = \sum_{\ell(x)\leq j} K_{x,w}[j-\ell(x)] = a_j(w) + \sum_{\ell(x)<j} K_{x,w}[j-\ell(x)], \tag{2}$$

for any $0 \leq j \leq \ell(w)$.

$(a) \Rightarrow (b)$ Suppose that $b_{k_0}(w) \neq 0$ and $b_k(w) = 0$, for each $0 < k < k_0$. Then, by (ii), the coefficient of q^k in $K_{x,w}$ is zero, for each $0 < k < k_0$, and any $x \leq w$. Hence,

$$c_k(w) = \sum_{\substack{x\leq w \\ \ell(x)=k}} q^{\ell(x)} = a_k(w).$$

Furthermore,

$$c_{k_0}(w) = \sum_{\substack{x\leq w \\ \ell(x)=k_0}} q^{\ell(x)} + b_{k_0}(w) = a_{k_0}(w) + b_{k_0}(w) > a_{k_0}(w)$$

by assumption and the fact that $b_i(w) \geq 0$, for each i.

$(b) \Rightarrow (c)$ If $a_k(w) = c_k(w)$ then Theorem 1.1 gives

$$\alpha_{\ell(w)-k}(w) \geq a_k(w) = c_k(w) = c_{\ell(w)-k}(w) \geq a_{\ell(w)-k}(w)$$

so that $a_k(w) = a_{\ell(w)-k}(w)$. Equation (2) and the assumption $a_k(w) = c_k(w)$ implies that $b_k(w) = 0$, for each $0 < k < k_0$. Also, $a_{k_0}(w) < c_{k_0}(w)$ and monotonicity gives

$$0 < \sum_{\ell(x)<k_0} K_{x,w}[k_0 - \ell(x)] = b_{k_0}(w).$$

To obtain the result it suffices to show that $a_{\ell(w)-k_0}(w) = c_{\ell(w)-k_0}(w)$. Suppose this is not the case, so that

$$\sum_{\ell(x)<\ell(w)-k_0} K_{x,w}[\ell(w) - k_0 - \ell(x)] > 0.$$

It is well-known that $\deg K_{x,w}(q) \leq (\ell(w) - \ell(x) - 1)/2$ [KL79] so that those $x \leq w$ that can contribute to the sum above must satisfy $\ell(w) + 1 - 2k_0 \leq \ell(x) < \ell(w) - k_0$. Thus, there is some $x \leq w$ satisfying this constraint such that $K_{x,w}[\ell(w) - k_0 - \ell(x)] \neq 0$. However, any such x must have $\ell(w) - k_0 - \ell(x) \in \{1, \ldots, k_0 - 1\}$, which contradicts monotonicity and the fact that $b_k(w) = 0$, for any $0 < k < k_0$.

$(c) \Rightarrow (a)$ This argument is similar to the previous argument. Let $0 < k_1 < k_0$ be a minimal index such that $b_{k_1}(w) > 0$, and suppose $a_k(w) = a_{\ell(w)-k}(w)$, for $0 < k < k_0$. Then,

$$a_{k_1}(w) + b_{k_1}(w) = c_{k_1}(w) = c_{\ell(w)-k_1}(w) = a_{\ell(w)-k_1}(w) + C,$$

where

$$C = \sum_{\substack{x \leq w \\ \ell(x) < \ell(w) - k_1}} K_{x,w}[\ell(w) - \ell(x) - k_1] > 0.$$

As above we can find some $x \leq w$ with $\ell(w) + 1 - 2k_1 \leq \ell(x) < \ell(w) - k_1$ and such that $K_{x,w}[\ell(w) - \ell(x) - k_1] \neq 0$. However, by monotonicity, this contradicts the minimality of k_1.

□

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