SINGULAR LAGRANGIAN MANIFOLDS AND THEIR LAGRANGIAN MAPS

A. B. Givental'

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The paper examines the singularity theory of Lagrangian manifolds and its connection with variational calculus, classification of Coxeter groups, and symplectic topology. We consider the application of the theory to the problem of going past an obstacle, to partial differential equations, and to the analysis of singularities of ray systems.

INTRODUCTION

The notion of Lagrangian manifold plays a central role in all applications of symplectic geometry — in variational calculus and classical mechanics, in representation theory and quantization, in the theory of hyperbolic differential equations and the geometry of spaces with singularities.

A Lagrangian manifold, by definition, is a submanifold of medium dimension in a symplectic manifold on which a symplectic structure (i.e., a closed nondegenerate differential 2-form) identically vanishes. A basic example of a symplectic manifold is the tangent fibration T^*B with canonical symplectic structure (see [1, p. 171]). The graph of the differential of a function on B is a Lagrangian section in T^*B. This example is the basis of Weinstein's definition of generalized function as an arbitrary Lagrangian submanifold in T^*B [44]: it can be considered as the graph of the differential of a (generally) multivalued function on B. This is indeed the approach that we adopt in our paper.

Multivalued functions naturally arise in variational calculus as solutions of the Hamilton—Jacobi equations. The solution of the Cauchy problem for the Hamilton—Jacobi equation, while locally single-valued, becomes many-valued under analytic continuation. The branching points form a caustic — the envelope of the corresponding bundle of extremals of the variational problem.

Systematic analysis of singularities of Lagrangian maps, i.e., projections of Lagrangian submanifolds in T^*B to the base of the Lagrangian fibration $T^*B \to B$, was begun by Arnol'd [2] in 1972. For example, the description of the multivaluedness of the distance function from a Riemannian manifold B to a given submanifold is reducible to such a problem. One of the results of [2] is the discovery that singularities of Lagrangian maps of nonsingular manifolds are classified by the degenerations of the critical points of functions and that the discrete part of their classification — the so-called simple singularities — naturally fits in the list of the crystallographic Coxeter groups.

In 1978, following the work on singularities of distance functions to submanifolds with an edge [3], Arnol'd extended the list of simple singularities by adding other crystallographic groups. Simple singularities corresponding to the noncrystallographic groups $I_2(5)$, H_3, H_4 were subsequently encountered in the problem of going past an obstacle (see [8, 31, 32]), although other simple singularities were also discovered. Thus, almost all irreducible Coxeter groups (except the symmetry group of regular n-gons with $n \geq 6$) were found to be connected with simple singularities in different variational problems. This connection of variational calculus with regular polygons remains a puzzle, despite the following theorem that we prove below.

THEOREM (see Sec. 9). In the class of Lagrangian manifolds locally diffeomorphic to the Cartesian product of plane curves, simple singularities of multivalued functions are in one-to-one correspondence with finite irreducible Coxeter groups. Specifically, the germ of a multivalued function is simple if and only if its graph is locally diffeomorphic to the discriminant of one of these groups or to the product of the discriminant and a nonsingular manifold.

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The Lagrangian manifolds in this theorem in general have singularities. In Chap. 1 we construct a general theory of singular Lagrangian manifolds and their Lagrangian maps. Section 1 proves an analog of Darboux's theorem, and Sec. 3 presents a stability criterion of Lagrangian maps. The property of infinitesimal versality of a family of functions can be stated in terms of the Lagrangian map generated by this family. In this form, the property is carried over to the Lagrangian maps of a singular manifold, adding a necessary and sufficient condition of their stability. The basic geometrical properties of bifurcation diagrams are preserved in this more general setting (see Sec. 4).

The interest in singular Lagrangian manifolds is attributable primarily to their occurrence in problems of going past an obstacle, i.e., in the analysis of singularities of the distance function on a Riemannian manifold with an edge (Sec. 6). Thus, the rays tangent to the edge on the Euclidean plane form the bundle of extremals in this problem. Therefore the system of rays is the Lagrangian curve projectively dual to the edge. The inflection points of the edge correspond to semicubic cusp points of the ray system. This example is the beginning of the series of singularities of Lagrangian manifolds that we call open swallowtails (5.1). They have an important property of universality in relation to the Whitney projections of integral manifolds (see Sec. 8) and occur in many problems, not necessarily as Lagrangian manifolds (see 5.2, 8, 11, and also [5, 26, 30]). In Sec. 10 we determine all the simple singularities of Lagrangian maps of open swallowtails. The list of the corresponding multivalued functions coincides with the list of singularities discovered by Shcherbak in the problem of going past an obstacle [32].

A complete classification of stable singularities of Lagrangian maps in any variational problem includes a classification of the critical points of functions, and is therefore intractable. However, the accumulated experience shows that the singularities of the mapped Lagrangian manifolds occurring in this variational problem are explicitly enumerable. The technique developed in this paper (Sec. 3) in principle makes it possible, for given singularities of Lagrangian manifolds, to advance as far as desired with the classification of their Lagrangian maps (insofar as this is feasible for the classification of singularities of functions).

In Sec. 11 we again apply this technique to study the simplest singularities of the isotropic maps $\mathbf{R}^n \to T^*\mathbf{R}^n$. In 11.4 we describe the contribution of these singularities to the topology of closed Lagrangian surfaces ($n = 2$). The connections of symplectic topology with singularity theory are also considered in Sec. 2, which formulates a local version of the problem of Lagrangian self-intersections.

For the sake of simplicity, all objects and morphisms considered in the body of the paper are assumed real analytic or complex analytic. In fact, all real analytic results remain also valid in the C^∞-category. Appendix 2 explains how to accomplish this extension.

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Chapter 1: GENERAL THEORY

1. RELATIVE DARBOUX THEOREMS

We extend the classes of Lagrangian and Legendrian manifolds to include singular manifolds.

Definition. A Lagrangian (Legendrian) manifold is an analytic subset of the symplectic (contact) manifold which in the neighborhood of each of its nonsingular points is a Lagrangian (Legendrian) submanifold in the ordinary sense.

In particular, a Lagrangian (Legendrian) manifold is of the same dimension at all its nonsingular points and the set of these points is dense in the manifold. When we say that a differential form vanishes on a singular manifold or that a vector field is tangent to a singular manifold, we mean that this is so at all nonsingular points.

Let Λ be an analytic subset in the manifold M. The differential forms on M that vanish on Λ form a subcomplex in the de Rham complex of the manifold M.

The factor-complex $(\Omega^*(\Lambda), d)$ will be called de Rham complex and its cohomology spaces $H^*(\Lambda)$ will be called de Rham cohomology spaces of the subset Λ.

In our applications, conditions of the form $H^i(\Lambda) = 0$ will be checked using the following lemma due to Poincaré.

Poincaré Lemma. The de Rham complex of a quasihomogeneous analytic set with positive weights is acyclic.

Proof. Let E be the Euler field of quasihomogeneous stretchings in the space M tangent to the analytic subset Λ, α a k-form in M closed on Λ, $\alpha=\sum_{s>0}\alpha_s$ its decomposition into quasihomogeneous terms, $\beta=\sum_{s>0}\alpha_s/s$. Then $\alpha = di_E\beta$ in the complex $\Omega^*(\Lambda)$. This follows from Cartan's formula $s\alpha_s=L_E\alpha_s=i_Ed\alpha_s+di_E\alpha_s$ and the equalities $d\alpha_s|_\Lambda = 0$, which hold for all s.

Remark 1. Positive quasihomogeneity should be regarded as an analytic analog of contractibility of the set Λ. It can be shown (see Sec. 2) that positive quasihomogeneity (in appropriate coordinates) of a germ of a plane curve is equivalent to acyclicity of its de Rham complex. On the other hand, any germ of a plane holomorphic curve is contractible in the homotopic sense [11].

Let Λ be a Lagrangian manifold in the symplectic space M ($\simeq C^{2n}$ or R^{2n}). The class $[\alpha] \in H^1(\Lambda)$ is well defined by the symplectic 2-form $\omega = d\alpha$ and is called its characteristic class.

Fix the pair (M, Λ), where Λ is an analytic subset in M. The symplectic structures in M relative to which Λ is a Lagrangian manifold will be called equivalent if one is carried to another by a diffeomorphism M that preserves Λ.

The characteristic class of a symplectic form is obviously an invariant of the form's equivalence class.

We will use the following stability terminology. A germ is called stable if its representative is locally equivalent to any small perturbation. We do not require that this equivalence preserve the application point of the germ. However, two germs are called equivalent if there exists an equivalence of their representatives that carries one application point into another. In particular, a germ close to a stable germ is not necessarily equivalent to the latter.

THEOREM 1. A germ of a symplectic structure in M relative to which $\Lambda \subset M$ is a Lagrangian manifold is stable in its characteristic class.

Proof. We will prove that two close germs ω_0, ω_1 of symplectic structures on M that vanish on Λ and have the same characteristic class are mapped to one another by a germ of a diffeomorphism that is close to the identity diffeomorphism and preserves Λ. Consider the family $\omega_t=(1-t)\omega_0+t\omega_1,\ 0\leqslant t\leqslant 1$ of germs of symplectic structures. We seek a family g_t of germs of diffeomorphisms that take ω_t to ω_0: $g_t{}^*\omega_t=\omega_0,\ g_0=\mathrm{id},\ g_t\Lambda=\Lambda$. Differentiating, we obtain the equivalent equation

$$L_{V_t}\omega_t+\omega_1-\omega_0=0, \tag{1}$$

where V_t is the velocity field of the diffeomorphisms g_t. Since the characteristic classes of ω_0 and ω_1 coincide, then $\omega_0 - \omega_1 = d\alpha$, where $\alpha|_\Lambda = 0$. Therefore (1) follows from

$$i_{V_t}\omega_t=\alpha. \tag{2}$$

LEMMA. The map $V\mapsto i_V\omega$ is an isomorphism of the space of vector fields tangent to the Lagrangian manifold $\Lambda \subset (M, \omega)$ to the space of 1-forms vanishing on Λ.

This obvious lemma implies unique solvability of Eq. (2) in the class of vector fields tangent to Λ.

THEOREM 1′. A germ of a contact structure in N relative to which $\Lambda \subset N$ is Legendrian is stable in the class of such contact structures.

The proof uses the symplectization functor (see [10]). The symplectization of the contact manifold N is a symplectic manifold M whose points are nonzero covectors on N that vanish on the hyperplane of the contact field at the application point of the covector. The projection π: M → N associating an application point to a covector is a one-dimensional fibration. The fibers are the orbits of the action of the group G of nonzero scalars defined by multiplication of a covector by a number. The 1-form α taking the value $p(\pi_*v)$, $p \in T_{\pi(p)}{}^*N$, on the vector $v \in T_pM$ is canonically defined on M. The symplectic structure $\omega - d\alpha$ is homogeneous of degree 1 relative to the action of the group G, and together with this action it defines a contact structure on N = M/G. The contact geometry on N is transformed by symplectization into a G-homogeneous symplectic geometry on M. In particular, we have the correspondence $\Lambda\mapsto\pi^{-1}(\Lambda)$ between Legendrian manifolds in N and G-invariant Lagrangian manifolds in M.

In proving Theorem 1′, we assume that two homogeneous 1-forms α_0, α_1 vanishing on $\pi^{-1}(\Lambda)$ and the family of symplectic structures $\omega_t=(1-t)d\alpha_0+td\alpha_1.$ are defined on the G-fibration π: M → N. In Eq. (2), $\alpha = \alpha_1 - \alpha_0$ vanishes on $\pi^{-1}(\Lambda)$. Since α and ω_t are homogeneous of degree 1, the solution V_t of Eq. (2) is homogeneous of degree 0. Therefore, the diffeomorphisms g_t commute with the G-action and take ω_t to ω_0 while preserving $\pi^{-1}(\Lambda)$ and the fibration.

Omitting these diffeomorphisms in N, we obtain diffeomorphisms that take the corresponding contact structures into one another while preserving Λ.

Remark 2. The homotopic method of proof of Theorems 1 and 1′ goes back to the work of Moser [39] on volume elements. We can similarly prove that the symplectic (contact) type of a germ of a nonsingular submanifold in a symplectic (contact) space is determined by the restriction of the symplectic (contact) structure to this submanifold. For such proof and discussion of results, see [10, Ch. 2, Sec. 1].

An intermediate result is available for the germs of singular coisotropic manifolds with $H^1 = 0$: for a fixed "interior" geometry, the "exterior" geometry is stable. Results of this type will be called relative Darboux theorems, because the stability condition of the symplectic (contact) structure in these theorems is independent of the structure itself.

As we shall now see, not every singular manifold admits a Lagrangian embedding.

A germ of an analytic set is called reduced if it cannot be decomposed into the product of a germ of an analytic set and a nonsingular manifold of positive dimension.

Let $\mathcal{O}_\Lambda = \Omega^0(\Lambda)$ be the algebra of germs of analytic functions on Λ, $\mathfrak{m} \subset \mathcal{O}_\Lambda$ a maximal ideal. Then $d(\Lambda) = \dim \mathfrak{m}/\mathfrak{m}^2$ is the least dimension of the space in which Λ is embedded.

Proposition 1. For a reduced germ of a Lagrangian manifold Λ, $d(\Lambda) = 2\dim \Lambda$.

Proof. If $d(\Lambda) < 2\dim \Lambda$, then there exists a germ of an analytic function that vanishes on Λ whose differential does not vanish at the origin. The Hamiltonian flow of this function fibers Λ into nonsingular curves over a Lagrangian germ whose dimension is less by 1. Therefore Λ is not reduced.

COROLLARY. A reduced germ of an n-dimensional analytic set of codimension $<n$ does not have a Lagrangian embedding in a 2n-dimensional symplectic space.

The following proposition shows that in the complex case a reduced germ Λ with $H^1(\Lambda) = 0$ admits at most one (up to equivalence) Lagrangian embedding.

Proposition 2. Let Λ be a reduced germ of a Lagrangian manifold. Then in each characteristic class there are at most a finite number (one in the complex case) of classes of equivalent symplectic structures in which Λ or its product $\Lambda \times D$ with a nonsingular manifold is Lagrangian.

Proof. The values at the origin of the coordinate 2-forms that vanish on Λ and have a given characteristic class in $H^1(\Lambda)$ constitute a finite-dimensional space. Degenerate 2-forms constitute a proper algebraic hypersurface in this space. Symplectic structures whose values at the origin lie in one connected component of the complement of this hypersurface are equivalent by Theorem 1 and by the fact that the diffeomorphism of a reduced germ preserves the origin.

For the product $\Lambda \times D$, repeating dim D times the construction from the proof of Proposition 1, we obtain a decomposition of the symplectic space $(M, \omega) \supset \Lambda \times D$ into the product of $(M', \omega') \supset \Lambda$ and $(M'', \omega'') \supset D$. Hence it follows that the equivalence class of the symplectic form ω depends only on the first factor. Q.E.D.

For the product $\Lambda \times D$ of the reduced Lagrangian germ Λ and the nonsingular manifold D, $d(\Lambda \times D) = 2\dim \Lambda + \dim D$ and, conversely, this equality and the Lagrangian property of $\Lambda \times D$ imply that the germ Λ is Lagrangian. In the following example, the reduced factor Λ of the Lagrangian germ $\Lambda \times D$ does not have Lagrangian embeddings: $d(\Lambda) > 2\dim \Lambda$.

Example. Let $\Lambda \subset \mathbf{C}^2$ be a germ at zero of the Lagrangian curve with the equation $x^3 + y^7 + axy^5 = 0$. For $a \neq 0$, $\dim H^1(\Lambda) = 1$ and the 1-form xdy generates $H^1(\Lambda)$. The function $u = \int_\Lambda x\,dy$ is analytic on the resolution $\tilde{\Lambda}$ of the germ Λ. The image $\tilde{\Lambda} \xrightarrow{(u,x,y)} \mathbf{C}^3$ is a Legendrian curve L relative to the contact structure $du = x\,dy$. We have $H^1(L) = 0$, $d(L) = d(\Lambda) + 1 = 3$. The symplectization of L is a Lagrangian surface in $\mathbf{C}^\times \times \mathbf{C}^3$ diffeomorphic to $\mathbf{C}^\times \times L$. Its germs at the points of the edge $\mathbf{C}^\times \times \{0\}$ have $d = 4$ and an acyclic de Rham complex. They are pairwise nonisomorphic as germs of Lagrangian manifolds: diffeomorphisms that preserve $\mathbf{C}^\times \times L$ and take the symplectic structure ω to $\lambda\omega$ move the application point of the germ $(z, 0)$ to the point $(\lambda^{-1}z, 0)$.

2. LOCAL SYMPLECTIC TOPOLOGY

The following conjecture of symplectic topology was recently proved by Gromov [35]:

a nonsingular closed manifold does not have exact Lagrangian embeddings in a space with a standard symplectic structure.

Recall that a Lagrangian embedding is called exact if the symplectic form has characteristic class 0.

For curves, this theorem implies that a closed plane curve enclosing a zero area is self-intersecting. Equally obvious is the corresponding assertion for a closed hypersurface enclosing a zero volume. However, for Lagrangian manifolds of dimension ≥ 2 the problem is far from trivial and remained open for a long time.

This problem has the following local analog that relates to singular Lagrangian manifolds.

Conjecture. If $H^n(\Lambda) \neq 0$ for the germ Λ of a n-dimensional Lagrangian manifold, then $H^1(\Lambda) \neq 0$ and the characteristic class of the symplectic structure is nonzero.

We will give a heuristic "proof" of the conjecture. Assume that the characteristic class of the symplectic structure ω is zero. Then $\omega = d\alpha$, where $\alpha|_\Lambda = 0$. Let V be the vector field tangent to Λ defined by the equation $i_V\omega + \alpha = 0$. The flow of the field V consists of canonical transformations of valency -1: $L_V\omega = -\omega$. We may take $V(0) = 0$. The spectrum of the linearization of the field V at zero is symmetric about $-1/2$. If the field V is equivalent to its linear part, then it flow "analytically contracts" Λ to an isotropic subspace or to a proper analytic subset in it. Therefore Λ should have homological dimension $<$n.

THEOREM 2 (A. N. Varchenko, A. B. Givental'). The conjecture is true for $n = 1$.

Proof. The germ of a singular plane curve is defined by the equation $f = 0$, where $f \in \mathcal{O}_{\mathbf{C}^2}$ has an isolated critical point at zero. The integrals of the holomorphic 1-form α over vanishing cycles on the curves $f = t$ are series expandable in fractional powers of t and logarithms (see [14]). The exponent $\lambda(\alpha)$ of the form α is the least exponent λ of the monomials t^λ, $t^\lambda \ln t$ in these series. We know [15] that $\lambda(\alpha) > 0$ and that the least exponent is observed precisely for those 1-forms α for which the 2-form $d\alpha$ is nondegenerate. We will show that $[\alpha] = 0 \Rightarrow H^1(f^{-1}(0)) = 0$ for a nondegenerate form $d\alpha$. We may take $\alpha|_\Lambda = 0$. Then $df \wedge \alpha = f\delta$, where $\delta \in \Omega^2_{\mathbf{C}^2}$. Let $\delta = d\beta$. Then

$$\int_{\Gamma_t \subset f^{-1}(t)} \alpha = t \frac{d}{dt} \int_{\Gamma_t} \beta.$$

Hence it follows that the exponent $\lambda(\beta) = \lambda(\alpha)$ is minimal, and therefore δ is a nondegenerate 2-form: $\delta = \varphi dx \wedge dy$, $\varphi(0) \neq 0$. The equality $\varphi^{-1} df \wedge \alpha = f dx \wedge dy$ implies that f lies in its gradient ideal: $f \in (f_x, f_y)$. By Saito's theorem [41], the function f is quasihomogeneous in appropriate coordinates, and $H^1(f^{-1}(0)) = 0$ by the Poincaré lemma.

Remark 3. a) The same argument is applicable to the germs of hypersurfaces at an isolated singular point. However, the assertions of minimality of the exponents have not yet been proved in full generality [15]. Moreover, a hypersurface of dimension >1 may have nonisolated singularities. Therefore, the analog of Theorem 2 for volume forms remains a conjecture.

b) Our conjecture is apparently true for nonsingular Lagrangian manifolds in a standard complex symplectic space under certain conditions of nonsingularity at infinity, replacing the closure property of the Lagrangian manifold in the real case. For instance, on a $\mathbf{C}^2$ nonsingular algebraic curve of genus > 0 transversal to the straight line at infinity, the characteristic class of the symplectic form is nonzero. This can be proved by embedding this curve in a versal deformation of a homogeneous singularity and applying the Gauss–Manin connectivity properties (see [14]) in the fibration of one-dimensional cohomologies.

3. LAGRANGIAN MAPS

3.1. *Definitions.* A Lagrangian map is the diagram $\Lambda \hookrightarrow (M, \omega) \to B$ that consists of an embedding of the Lagrangian manifold Λ in a symplectic manifold (M, ω) and a Lagrangian fibration $M \to B$. Equivalence of Lagrangian maps is the commutative diagram

$$\begin{array}{ccccc} \Lambda & \subset & (M,\omega) & \to & B \\ \updownarrow & & \updownarrow & & \updownarrow \\ \Lambda' & \subset & (M',\omega') & \to & B', \end{array}$$

where the middle arrow is symplectomorphism.

A Lagrangian map is called stable if any close Lagrangian map (i.e., a Lagrangian map obtained from the original map by a small change of the symplectic structure and/or the Lagrangian fibration) is carried to the given map by a symplectomorphism close to identity.

We define the graph of the Lagrangian map $\Lambda \hookrightarrow T^*B \xrightarrow{\pi} B$ with $H^1(\Lambda) = 0$ in the following way. The 1-form of the action α on T^*B is the differential of the function φ in the complex $\Omega^*(\Lambda)$. The graph Φ of a Lagrangian map is defined as the image of the map $\Lambda \to \mathbf{C} \times B : \lambda \mapsto (\varphi(\lambda), \pi(\lambda))$. The graph in general is a hypersurface. It may be regarded as the graph of a multivalued function on B — the generating function of the Lagrangian manifold Λ. Equivalence of Lagrangian maps acts on this function as change of variables and addition of a single-valued function. The graph defines the original Lagrangian map: the Lagrangian manifold Λ is the closure in T^*B of the 1-graph of the generating function. Suspension and restriction of a Lagrangian map to the submanifold $B' \subset B$ are defined resp. by adding new variables on which the generating function does not depend explicitly and by restricting the generating function to B'.

Partition the points of the Lagrangian manifold Λ into the equivalence classes of the germs of the Lagrangian map $\Lambda \hookrightarrow (M, \omega) \to B$ at these points. The modality of a stable Lagrangian map is the number of continuous parameters needed for parametrization of these classes. A Lagrangian map is called simple if its modality is 0, i.e., if it is stable and its germs at all points of the Lagrangian manifold belong to a finite number of equivalence classes.

3.2. We define the local algebra Q of the germ of the Lagrangian map $\Lambda \overset{i}{\hookrightarrow} (M, \omega) \overset{\pi}{\to} B$ as the algebra of functions on $\Lambda \cap \pi^{-1}(0)$: $Q = \mathcal{O}_\Lambda / \mathcal{O}_\Lambda (\pi i)^* \mathfrak{m}_B$. If dim Q $< \infty$, then the germ is said to be of finite multiplicity, and dim Q is its multiplicity.

There is an affine structure in the fiber F = $\pi^{-1}(0)$ of a Lagrangian fibration. Therefore the algebra $\mathcal{O}_F = \overset{\infty}{\underset{k=0}{\oplus}} \mathcal{O}_F^{(k)}$ of the fibration is graded by degrees of homogeneity of the functions. We have the map $\mathcal{O}_F \to Q$ that associates to a function on F its "restriction" to $\Lambda \cap$ F. The germ of a Lagrangian map is called (mini)versal if the map $\mathcal{O}_F^{(0)} \oplus \mathcal{O}_F^{(1)} \to Q$ is an (iso)epimorphism, i.e., if the local algebra is generated by linear inhomogeneous functions in a fiber of the Lagrangian fibration. Let (p, q) be the Darboux coordinates in the Lagrangian fibration. From the Weierstrass preparation theorem applied to the $\mathcal{O}_B$-module $\mathcal{O}_\Lambda$, we have

Proposition 3. Any function φ(p, q) on a Lagrangian manifold of a versal Lagrangian map is representable as a linear function inhomogeneous in the momenta,

$$\varphi(p, q) \equiv A_0(q) + A_1(q) p_1 + \ldots + A_n(q) p_n,$$

and conversely.

Remark 4. If $\mathfrak{m}_\Lambda \subset \mathcal{O}_\Lambda (\pi i)^* \mathfrak{m}_B$, then the Lagrangian map is versal: $\mathfrak{m}_\Lambda / \mathfrak{m}_\Lambda^2$ is generated by the functions $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ and is epimorphically mapped to the maximal ideal in Q, so that $(q_1, \ldots, q_n)$ lie in the map kernel. This versality criterion is useful because it is independent of the symplectic structure and only utilizes the properties of the projection $\Lambda \to$ B.

THEOREM 3. The germ $\Lambda \hookrightarrow (M, \omega) \to B$ of a Lagrangian map is stable if and only if it is versal and $H^1(\Lambda) = 0$.

Proof. By Theorem 1, the condition $H^1(\Lambda) = 0$ is necessary and sufficient for stability of the symplectic structure ω. We will show that versality of the Lagrangian map is necessary and sufficient for its stability relative to changes in the fibration for a fixed structure ω. A stable Lagrangian map is infinitesimally stable, i.e., every Hamiltonian vector field in M is representable as the sum of a Hamiltonian field tangent to Λ and a Hamiltonian field whose flow preserves the Lagrangian fibration. For the corresponding Hamiltonians this leads to the equality

$$\varphi = h + A_0(q) + A_1(q) p_1 + \ldots + A_n(q) p_n,$$

where φ is an arbitrary function from $\mathcal{O}_M$ and h vanishes on Λ. This implies versality. Conversely, assume given a family of Lagrangian maps $\theta_t : \Lambda_t \hookrightarrow (M, \omega) \to B$ obtained from the versal map θ_0 by a family of symplectomorphisms g_t: M $\to$ M: $\Lambda_t = g_t \Lambda_0$. Construct a germ of the lagrangian map $\Xi : \Lambda \times \mathbb{R} \hookrightarrow (M, \omega) \times T^*\mathbb{R} \to B \times \mathbb{R}$ whose restriction to B $\times$ {t} is θ_t. Let $(p_1, \ldots, p_n)$ be Darboux coordinates in T^*R. The Lagrangian map Ξ is versal, and its local algebra Q_Ξ coincides with Q_{θ_0} and is generated by the functions $(1, p_1, \ldots, p_n)$. By Proposition 3, . $p_0 \equiv A_0(q, t) + A_1(q, t) p_1 + \ldots + A_n(q, t) p_n$ in $\mathcal{O}_{\Lambda \times \mathbb{R}}$. The function $h = p_0 - A_0 - \sum A_i p_i \ (i \geqslant 1)$ vanishes on $\Lambda \times$ R, is linear in the momenta, and {h, t} = 1. Therefore, the flow of the Hamiltonian h restricted to the hypersurface $h^{-1}(0)$ carries θ_0 to θ_t in time t. Q.E.D.

nian h restricted to the hypersurface $h^{-1}(0)$ carries θ_0 to θ_t in time t. Q.E.D.

Let $\Lambda \hookrightarrow (M, \omega)$ be a germ of a Lagrangian manifold. We say that two Lagrangian maps defined by the Lagrangian fibrations (M, ω) $\to$ B and (M, ω) $\to$ B′ have the same k-jet if the Lagrangian fibrations are transformed to one another by a symplectomorphism with the identity k-jet.

COROLLARY 1 *(sufficient jet theorem).* Let the germ $\Lambda \hookrightarrow (M, \omega) \to B$ of a Lagrangian map be versal and let the k-th power of the maximal ideal $\mathfrak{m}_Q$ in its local algebra Q be zero. Then the germ of the Lagrangian map $\Lambda \hookrightarrow (M, \tilde{\omega}) \to B'$ with the same k-jet is equivalent to this germ.

Example. The 1-jet at zero of the versal Lagrangian map $(p, q) \mapsto q$ of the plane curve $q = p^2$ is not sufficient. This means that the order k of the sufficient jet in Corollary 1 in general cannot be lowered.

Proof. The germ of a symplectomorphism with the identity k-jet may be deformed into the identity germ in this class of symplectomorphisms. Therefore Corollary 1 will follow from Theorem 3 if we show that the germ $\Lambda \hookrightarrow (M, \omega) \to B'$ is versal.

Let (p, q) be the Darboux coordinates in the fibration (M, ω) $\to$ B. If $\varphi(p, q) \in \mathfrak{m}_M^{k+1}$, then $\varphi \in \mathfrak{m}_\Lambda I$, where I is the ideal in $\mathcal{O}_\Lambda$ generated by $q_1, \ldots, q_n$. Indeed, a monomial of degree k in the variables $p_1, \ldots, p_n$ defines the zero class in $Q = \mathcal{O}_\Lambda / I$, i.e., it lies in I.

Let $P_i = p_i + R_i(p, q)$, $Q_i = q_i + S_i(p, q)$ be the components of the symplectomorphism carrying $(M, \omega) \to B$ to $(M, \omega) \to B'$, $R_i, S_i \in \mathfrak{m}_M^{k+1}$. Then the ideal I′ in $\mathcal{O}_\Lambda$ generated by $Q_1, \ldots, Q_n$ lies in I. On the other hand, $I \subset \mathfrak{m}_\Lambda I + I'$. Hence, I = I′, by Nakayama's lemma [9] applied to the $\mathcal{O}_\Lambda$-module I/I′. Since $R_i \in I$, then the functions $1, P_1, \ldots, P_n$ generate the space $Q = \mathcal{O}_\Lambda / I'$ if $1, p_1, \ldots, p_n$ generate $Q = \mathcal{O}_\Lambda / I$.

COROLLARY 2. The 1-jet of a versal Lagrangian map of a reduced Lagrangian manifold is sufficient.

Proof. If $\varphi \in \mathfrak{m}_M^2$, then $\varphi \in \mathfrak{m}_\Lambda I$. Indeed, in the expansions $p_i p_j = \sum a_{ij}^k(q) p_k + b_{ij}(q)$ $a_{ij}^k \in I$, $b_{ij} \in I^1$, because a Hamiltonian that vanishes on a reduced Lagrangian manifold has zero 1-jet at the origin.

COROLLARY 3. Let $\Lambda \hookrightarrow (M, \omega)$ be a germ of an algebraic Lagrangian manifold in the standard symplectic space. Then a stable germ of the Lagrangian map $\Lambda \hookrightarrow (M, \omega) \to B$ is equivalent to the algebraic germ.

3.3. A Legendrian map is the diagram $\Lambda \hookrightarrow M \to B$ that consists of an embedding of the Legendrian manifold Λ in the Legendrian fibration M → B. The image of Λ in B is called the front of the Legendrian map. The front, in general, is a singular hypersurface in B. The Legendrian fibration M → B is locally canonically identified with the fibration $PT^*B \to B$ of contact elements on B. A typical Legendrian map is defined by its front: the Legendrian manifold Λ consists of the contact elements of the front (the nontypical Legendrian maps among all the Legendrian maps of the manifold Λ constitute a set of infinite codimension). Equivalence of these Legendrian maps is a diffeomorphism of their fronts.

To the Lagrangian map $\Lambda \hookrightarrow T^*B \to B$ corresponds the Legendrian map $\Lambda \hookrightarrow J^1B \to J^0B$ of the same manifold Λ if $H^1(\Lambda) = 0$: the graph $\Phi \subset J^0B$ of the Lagrangian map is the front of the Legendrian map. The fibers of the fibration $J^0B \to B$ are transversal to the contact elements to Φ. Replacing the vertical field $\partial/\partial u$ in J^0B with another vector field transversal to the contact elements to Φ defines another Lagrangian map $\Lambda \hookrightarrow T^*B' \to B'$. These Lagrangian maps are not necessarily equivalent (see 3.4), although the corresponding Legendrian maps coincide.

We define the local algebra Q of a germ of the Legendrian map $\Lambda \overset{i}{\hookrightarrow} PT^*B \overset{\pi}{\to} B$ at the point $0 \in \Lambda$ by setting

$$Q = \mathcal{O}_\Lambda / \mathcal{O}_\Lambda (\pi i)^* \mathfrak{m}_B.$$

In the projective space $PT^*_{\pi i(0)}B$ take an affine chart containing the point i(0) and a coordinate system $(p_1, \ldots, p_n)$ centered at this point. A germ of the Legendrian map is called (mini)versal if the functions $(1, p_1, \ldots, p_n)$ generate (form a basis in) Q. This condition is independent of the choice of the affine chart and the coordinate system.

The Legendrian map corresponding to a versal Lagrangian map is versal. The converse, in general, is not true.

The following theorem is proved like Theorem 3, applying Theorem 1′ instead of Theorem 1.

THEOREM 3′. A germ of a Legendrian map is stable if and only if it is versal.

Analogs of Corollaries 1 and 3 of Theorem 3 are also true in the Legendrian case.

3.4. Let $\Lambda \times D \hookrightarrow T^*B \to B$ be a germ of the Lagrangian map of the product of the Lagrangian germ Λ by a nonsingular manifold. Assume that the projection $\{0\} \times D \to B$ is an immersion. Then the restriction of a Lagrangian map to the transversal $B_0 \subset B$ to the image of $\{0\} \times D$ is a germ of the Lagrangian map $\Lambda \hookrightarrow T^*B_0 \to B_0$.

Proposition 4. If the restriction $\Lambda \hookrightarrow T^*B_0 \to B_0$ is stable, then the original germ of the Lagrangian map is equivalent to its suspension.

Proof. Restricting the original Lagrangian map Ξ to the fibers B_t, $t \in D$, of the tube neighborhood B → D of the image of $\{0\} \times D$ in B, we obtain a family of Lagrangian maps $\theta_t : \Lambda \hookrightarrow T^*B_t \to B_t$. The corresponding family of generating functions considered as a multivalued function on B is the generating function for Ξ. Since θ_0 is stable, then the family θ_t is equivalent to a constant family. Therefore, the generating function for Ξ reduces to a form independent of t.

As the degree of a germ of finite multiplicity of a Lagrangian map we take the degree of the projection Λ → B, i.e., the number of complex preimages of the general point. The degree does not exceed the multiplicity dim Q of the image, but may be less than dim Q (see Chap. 3, Sec. 3).

Let $\Lambda \hookrightarrow T^*B \to B$ be a germ of a Lagrangian map of degree μ, $H^1(\Lambda) = 0$, $u: \Lambda \to C$ its generating function. Let us define the Lyashko–Looijenga map from the base B to the space $\mathcal{P}_\mu$ of polynomials of degree μ of a single variable with a fixed highest order coefficient and zero sum of roots. This map associates to the point $q \in B$ a polynomial whose roots are the μ values of the generating function u at the points $\pi^{-1}(q)$ shifted by a constant so that their sum is zero. The image of the Lyashko–Looijenga map is a germ of an analytical subset in $\mathcal{P}_\mu$ — an invariant of the equivalence class of a germ of a Lagrangian map. If the dimension of this image is less than $\mu - 1 = \dim \mathcal{P}_\mu$, then the Lagrangian map is unstable: the image may be altered by a diffeomorphism of the graph $\Phi \subset C \times B$ close to the identity diffeomorphism. In particular, if $\mu > \dim \Lambda + 1$, then the Lagrangian map is unstable and has function moduli.

Proposition 5. Let the germs of a Lagrangian map of degree not exceeding n + 1 occur irremovably in the restrictions of some Lagrangian map to n-dimensional submanifolds. Then this Lagrangian map is not simple.

Proof. The condition implies that on the Lagrangian manifold Λ of the given Lagrangian map there is a submanifold l of codimension n at the points of which merge not more than n + 1 sheets of the projection $\pi: \Lambda \to B$. If the germ of the Lagrangian map at the point $x \in l$ is equivalent to the germs at close points of l, then there exist dim l vector fields on B independent at the point $\pi(x)$ that generate automorphisms of the germ of the Lagrangian map at the point x. The Lyashko–Looijenga map of this germ therefore has fibers of dimension $\geq$dim l and an image of dimension $\leq$n. By assumption, this is a map to the space of polynomials of dimensions >n. Hence follows instability of the Lagrangian map.

Example. A stable germ of a Lagrangian map of a plane curve is equivalent to one of the germs $\Lambda_r \hookrightarrow T^*\mathbf{C} \to \mathbf{C}$, where Λ_r is defined by the equation $p^2 = q^r$, r = 1, 2, 3,... . These germs are simple. Indeed, by Proposition 5, the degree of a stable germ does not exceed 2. By the Weierstrass preparation theorem, a two-sheeted branched covering of a disk is locally diffeomorphic to Λ_r. By the Poincaré lemma, $H^1(\Lambda_r) = 0$. By Remark 4, the germs of Lagrangian maps of degree 2 are versal and, by Theorem 3, stable. Because of connectivity of the space of germs of symplectic structures on a plane, Lagrangian maps of curves Λ_r with the same r are equivalent to one another.

The germs of a Lagrangian map in general position of the manifold $\Lambda_r \times D$ are also simple at almost all the points in $\{0\} \times D$. This follows from Proposition 4.

4. THE GEOMETRY OF BIFURCATION DIAGRAMS

This is the title of the paper of Lyashko [27] which studies bifurcation diagrams of germs of holomorphic functions. Many of their properties carry over to versal Lagrangian maps.

The deformation F(x, q), $q \in B$, of a germ of the function F(x, 0) is called a generating family of the Lagrangian map $\Lambda \hookrightarrow T^*B \to B$:

$$\Lambda=\{(p, q) \mid \exists x : F_x(x, q)=0,\ p=F_q(x, q)\}.$$

If the matrix $(F_{xx}, F_{xq})|_{(0,0)}$ is of maximum rank, then Λ is a nonsingular Lagrangian manifold. A germ of a Lagrangian map of a nonsingular Lagrangian manifold is stable if and only if its generating family is a R^+-versal deformation of the germ F(x, 0) (see [9]).

The properties of versal Lagrangian maps of nonsingular manifolds are translatable into the language of the singularity theory of functions. Versality of the Lagrangian map of a nonsingular manifold is equivalent to R^+-versality of its generating family. The local algebra Q becomes the local algebra $\mathcal{O}_x/(F_x(\cdot, 0))$ of the germ of the function $F(\cdot, 0)$. Multiplicity in this case is equal to the degree of the Lagrangian map and is called the Milnor number. The graph of a Lagrangian map is the discriminant or the bifurcation diagram of zeros in the base of the R-versal deformation of the germ $F(\cdot, 0)$. The following concept in the theory of Lagrangian maps corresponds to the bifurcation diagram of the functions of the family F.

The bifurcation diagram of a germ of a Lagrangian map of finite multiplicity $\Lambda \hookrightarrow (M, \omega) \to B$ is the set $\Sigma \subset B$ of points where the number of values of the generating function is less than the degree of the Lagrangian map. The bifurcation diagram is a germ of an analytic hypersurface. In general, it has three components: the image of the singular points of Λ under the projection $\Lambda \to B$, the set of critical values of the projection $\Lambda \to B$ (jointly they form the caustic of the Lagrangian map), and the Maxwell stratum at the points of which merge the different branches of the generating function.

Let $\Lambda \hookrightarrow (M, \omega) \to B$ be a miniversal Lagrangian mapping whose degree μ is equal to the multiplicity (and is thus equal to dim Λ + 1).

Also assume that $H^1(\Lambda) = 0$. Denote by u: $\Lambda \to \mathbf{C}$ its generating function, $u=\int \sum p_i dq_i$. By Proposition 3, function $\varphi \in \mathcal{O}_\Lambda$ is uniquely representable in the form $\varphi=\sum_{i=1}^{\mu} a_i p_i,\ p_\mu=-1,\ a_i \in \mathcal{O}_B$. Denote by W_i vector field $\sum_{j=1}^{\mu-1} w_{ij}(q)\, \partial/\partial q_j$ on B determined from the expansion $u^i=\sum_{j=1}^{\mu} w_{ij} p_j,\ i=1, \ldots, \mu-1$ and by V_i the vector field $\sum_{j=1}^{\mu} [v_{ij}(q)-\delta_{ij} q_\mu]\, \partial/\partial q_j$ on $\mathbf{C} \times B$ determined from the expansion $p_i u=\sum_{j=1}^{\mu} v_{ij} p_j,\ i=1, \ldots, \mu$. Here $q = (q_1, ..., q_{\mu-1})$ are the coordinates on B, q_μ is the coordinate on C.

THEOREM 4 (see [21, 22, 27, 34, 42)].

1) Each germ of a vector field in $C \times B$ is uniquely representable as the sum of a vector field with coefficients independent of q_μ and a vector field tangent to the graph $\Phi \subset C \times B$.

2) The vector fields in $C \times B$ tangent to the graph Φ form a free module over $\mathcal{O}_{C\times B}$ with the generators $V_1, \dots, V_\mu$.

3) Assume additionally that the points from B where three or more values of the generating function merge form a set of codimension ≥ 2. Then the vector fields in B tangent to the bifurcation diagram $\Sigma \subset B$ are liftable in $C \times B$ to vector fields tangent to the graph Φ and form a free $\mathcal{O}_B$-module with the generators $W_1, \dots, W_{\mu-1}$.

Remark 5. The additional condition in part 3 is a condition on the singularities of Λ: it is sufficient that at the general points of the stratum of singularities of codimension 1 the manifold Λ be representable as a two-sheeted branched covering of a polydisk. Without this additional assumption, part 3 does not hold, as we can see from the following example of a Lagrangian map defined by the graph

$$u(u-q_1^2)(u+(1+q_2)q_1^2)=0 \text{ in } C\times C^2.$$

It can be shown that this map is miniversal, $\Sigma \subset C^2$ is defined by the equation $q_1 = 0$, but the translations $(q_1, q_2) \mapsto (q_1, q_2+\text{const})$ are not liftable to diffeomorphisms of the graph.

Proof. 1°. We will show that the fields V_i are tangent to the graph Φ. At a nonsingular point x, Φ is the graph of the function $q_\mu = u(q)$, and $\partial u/\partial q_i |_x = p_i(x)$, where $p_i(x)$ is the value of the momentum p_i at the preimage of the point x on Λ. We have

$$L_{V_i}(q_\mu-u(q))|_x=v_{i\mu}(q)-\sum_{j=1}^{\mu-1} v_{ij}(q)\,p_j(x)+u(x)\,p_i(x)=0$$

from the definition of the field V_i.

2°. If a fiber of the Lagrangian fibration $(M, \omega) \to B$ intersects Λ at μ different points, then these points are affinely independent in the fiber. This follows from versality. Indeed, if these points are dependent, then the values of linear inhomogeneous functions at these points are connected by a universal relationship, which cannot hold for all functions. But this contradicts Proposition 3, which states that any function is expressible in terms of linear functions.

Hence it follows that a vector field in $C \times B$ independent of q_μ and tangent to Φ is the zero field. Indeed, the vector of such a field at the general point $x \in C \times B$ is parallel to $\mu = \dim (C \times B)$ hyperplanes tangent to the graph at μ points in $C_k \cap \Phi$. Since these hyperplanes are independent, the vector is zero.

3°. We will show that every vector field in $C \times B$ is uniquely representable as the sum of a field independent of q_μ and a combination of the fields V_i. Indeed, each element $\varphi \in \mathcal{O}_{C\times \Lambda}$ is uniquely representable in the form $\varphi = \Sigma a_i(q, q_\mu)p_i$. Represent φ in the form $\varphi=\psi(q, p, q_\mu-u)(q_\mu-u)+\theta(q, p)$ and expand ψ and θ in p_i:

$$\psi=\Sigma b_i(q, q_\mu)p_i,\quad \theta=\Sigma c_i(q)p_i.$$

Note that θ and c_i are defined uniquely. Using the equalities $p_i(u-q_\mu)=\Sigma(v_{ij}-\delta_{ij}q_\mu)p_j$, we obtain $\Sigma a_i\partial/\partial q_i+\Sigma b_i V_i=\Sigma c_i\partial/\partial q_i$, where the right-hand side is uniquely determined by the field with the coefficients a_i.

4°. Part 1 of the theorem follows from 3° and 1°. From 2° and 3° it follows that V_i generate the $\mathcal{O}_{C\times B}$-module of the fields tangent to Φ: by 2°, the terms independent of q_μ vanish for this field. We will show that the fields V_i are independent at each point outside Φ. This will immediately imply their independence in the $\mathcal{O}_{C\times B}$-module of the fields tangent to Φ.

Consider the determinant Δ whose rows are the components of the fields V_i. This determinant vanishes on Φ and is a polynomial of degree μ in q_μ with nonzero constant highest order coefficient. By the Weierstrass preparation theorem, these are the properties of an irreducible equation of Φ in $C \times B$. Therefore $\Delta = 0$ is such an equation.

5°. Part 3 on the lifting of fields follows from the general theorem of Lyashko [27] and is ensured by the additional condition. Now let $W=\sum_{i=1}^{\mu-1} a_i(q)\partial/\partial q_i$ be a vector field tangent to $\Sigma \subset B$, $\tilde{W}=W-c(q, q_\mu)\,\partial/\partial q_\mu$ its lifting in $C \times B$ to a field tangent to Φ. Then in $\mathcal{O}_\Lambda$ we have $\sum_{i=1}^{\mu-1} a_i(q)\,p_i=c(q, u)$. Divide by c the equation Δ of the graph Φ:

$$c=\sum_{i=1}^{\mu-1} q_\mu^i b_i(q)+a_\mu(q)+R(q, q_\mu)\,\Delta(q, q_\mu).$$

Then in $\mathcal{O}_\Lambda$ we have $\sum_{i=1}^{\mu-1} u^i b_i = \sum_{i=1}^{\mu} a_i p_i$. Since the expansion in momenta is unique, we obtain $W = \sum b_i W_i$. Moreover, the fields W_i are independent over $\mathcal{O}_{B'}$, because W = 0 implies $a_\mu = \sum_{i=1}^{\mu-1} q^i_\mu b_i$ on Φ, contradicting the fact that the degree of the Lagrangian map is μ. It remains to check that the fields W_i are tangent to Σ. The computations in 1° show that the fields $\tilde{W}_i = W_i - (q^i_\mu + w_{i\mu}(q))\,\partial/\partial q_\mu$ are tangent to Φ. Since Σ is invariantly defined by the projection Φ → B, and the flows of the fields $\tilde{W}_i$ commute with this projection, then W_i are tangent to Σ.

THEOREM 5 (see [27, 21]). Let $\Lambda \hookrightarrow (M, \omega) \to B$ be a germ of a holomorphic miniversal Lagrangian map. Assume that its degree μ is equal to the multiplicity, $H^1(\Lambda) = 0$, and that all μ values of the generating function are equal only at one point from B. Then the complement B\Σ of the bifurcation diagram is the Eilenberg–MacLane space K(G, 1) of a subgroup G of finite index in the group of braids of μ strings.

Remark 6. Assume that under the conditions of the theorem the Lagrangian map is quasihomogeneous, $s_1, \dots, s_\mu > 0$ are respectively the weights of $q_1, \dots, q_\mu$. Then the index of G in the braid group is $\mu! s_\mu{}^\mu / s_1 \dots s_\mu$. This follows from [37] and the proof of Theorem 5.

Proof. The Lyashko–Looijenga map (see the proof of Proposition 4) is a holomorphic map $\lambda : B \to \mathcal{P}_\mu$ of a $(\mu - 1)$-dimensional space. Since the preimage of the origin in $\mathcal{P}_\mu$ consists of a single point, then this map is finite, and thus proper. The preimage of the discriminant Δ in the polynomial space $\mathcal{P}_\mu$ is by definition the bifurcation diagram Σ ⊂ B. Outside the caustic, the map has a nondegenerate differential. This follows from the versality condition, as explained in 2° in the proof of Theorem 4. Therefore $\lambda\colon (B \setminus \Sigma) \to (\mathcal{P}_\mu \setminus \Delta)$ is a connected nonbranched finite-sheeted covering. We know [12] that the space $\mathcal{P}_\mu \setminus \Delta$ is the Eilenberg–MacLane space of the group of braids of μ strings, which proves the theorem.

Remark 7. In Theorems 4 and 5 we studied the properties of miniversal Lagrangian maps whose degree is equal to multiplicity. Such a Lagrangian map defines a plane affine connectivity on the base of a fibration with singularities on the caustic. This connectivity is specified by the three-index tensor $a_{ij}{}^k$ determined from the expansions $p_i p_j = \sum_{k=1}^{\mu} a^k_{ij} p_k$, $i, j = 1, \dots, \mu - 1$. Parallel translation of covectors from T^*B takes $\Lambda \subset T^*B$ to itself and is uniquely characterized by this property. Λ is Lagrangian if and only if the curvature of the connectivity vanishes. In the complex case, the holonomy group of this connectivity is a subgroup of the permutation group of the branches of the projection Λ → B. Under the assumption that two branches merge at general points of the caustic, this group is the product of the symmetric groups permuting the sheets of the projection Λ → B for each irreducible component Λ. Indeed, by Zariski's theorem, the holonomy group is generated by loops around nonsingular points of the caustic. Each loop generates a transposition of the branches of Λ or an identity permutation. If Λ is irreducible, then the corresponding group is transitive and is generated by transpositions. It is thus a complete symmetric group.

THEOREM 6 (see [27, 21]). Let Φ ⊂ C × B be the graph of a stable germ of a Lagrangian map. Then the vector field in C × B close to the vertical field ∂/∂u is carried to this vertical field by a diffeomorphism preserving Φ.

Proof. Let us rectify the vector field by a diffeomorphism g close to the identity diffeomorphism. Then gΦ ⊂ C × B is the graph of the Lagrangian map $\Lambda \hookrightarrow T^*B \to B$ close to the original map. Indeed, stability of the original Lagrangian map implies that $H^1(\Lambda) = 0$. Therefore the manifold $L \subset J^1B$ of contact elements to Φ is isomorphic to Λ. In particular, d(L) = d(Λ), so that the map $J^1B \to T^*B$ produces a monomorphic projection of the tangent space $(\mathfrak{m}_L/\mathfrak{m}_L^2)^*$. The same is true for the close manifold g_*L, and therefore the image of g_*L in T^*B is isomorphic to Λ. Finally, equivalence of close Lagrangian maps sends gΦ back to Φ, preserving the field ∂/∂u.

COROLLARY. Assume that a stable germ of a Lagrangian map is quasihomogeneous (not necessarily with positive weights, but the weight of the symplectic form is nonzero) and its graph Φ is reduced. Then the germ of a vector field transversal to Φ is reducible to the form ±∂/∂u by a diffeomorphism preserving Φ.

Proof. By stability, $H^1(\Lambda) = 0$ and $(1, p_1, \dots, p_n)$ generate the local algebra $Q = \mathcal{O}_\Lambda / I$. By quasihomogeneity, $u = \int \Sigma p_i dq_i = \Sigma (\deg q_i) p_i q_i / (\deg u) \in \mathfrak{m}_\Lambda I$. The vector field v transversal to Φ defines the Lagrangian map $\Lambda \hookrightarrow T^*B' \to B'$ with the same graph Φ. Its local algebra $Q' = \mathcal{O}_\Lambda / I'$ coincides with Q. Indeed, $I' \subset (u, q_1, \dots, q_n) = I$, $I \subset \mathfrak{m}_\Lambda I + I'$. By Nakayama's lemma [9] applied to the $\mathcal{O}_\Lambda$-module I/I', we have I' = I. Therefore the Lagrangian map connected with the field v is versal. The corollary now follows from Theorem 6, because the field v can be deformed into ±∂/∂u in the class of fields transversal to Φ.

5. SYMPLECTIC STRUCTURES IN THE POLYNOMIAL SPACE

In this section, we introduce two symplectic structures in the finite-dimensional space $\mathcal{P}_\mu$ of polynomials of odd degree $\mu = 2k - 1$.

5.1. We start with the space V^{N+1} of binary forms of odd degree $N = 2k + 1$. V^{N+1} is the space of the irreducible representation of the symplectic transformation group SL_2 of a plane. This space has a unique SL_2-invariant exterior 2-form (up to a numerical multiplier) [18, 19]. The space of binary forms in the variables (u, v) is identified with the space of the polynomials

$$\frac{a_0 x^N}{N!}+\frac{a_1 x^{N-1}}{(N-1)!}+\ldots+\frac{a_N x^0}{0!}$$

in the variable $x = u/v$. In these coordinates, the SL_2-invariant 2-form is given by

$$\omega=\sum_{i=0}^{k}(-1)^i da_i\wedge da_{N-i}.$$

The symplectic structure in $\mathcal{P}_\mu$ is obtained from ω by two-fold reduction. The upper triangular subgroup $\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}$ acts on the polynomials by translating the argument $x\mapsto x+t$ while preserving the hyperplane $a_0 = 1$ and the symplectic structure ω. The Hamiltonian of the translation group has the form

$$h=\frac{1}{2}\sum_{i=0}^{2k}(-1)^i a_i a_{2k-i}.$$

Symplectic reduction on the hyperplane $a_0 = 1$ "forgets" the free term a_N, which is equivalent to differentiation of the polynomial. We thus obtain a symplectic space of polynomials of the form

$$\frac{x^{2k}}{(2k)!}+a_1\frac{x^{2k-1}}{(2k-1)!}+\ldots+\frac{a_{2k}}{0!}.$$

The first reduction commutes with argument translation. The second reduction is performed on the hyperplane $h = 0$. This equation permits expressing the free term a_{2k} in terms of all the other coefficients, because $a_0 \equiv 1$. The reduction thus produces a symplectic space $\mathcal{P}_{2k-1}$ of the polynomials

$$\frac{x^{2k-1}}{(2k-1)!}+a_2\frac{x^{2k-3}}{(2k-3)!}+\ldots+\frac{a_{2k-1}}{0!}$$

with a fixed highest order coefficient and zero sum of roots.

Definition. An open swallowtail is an algebraic subset in the space $\mathcal{P}_{2k-1}$ generated by polynomials with a root of multiplicity $\geq k$.

Proposition 6 [18, 19]. An open swallowtail is a Lagrangian manifold in the symplectic space $\mathcal{P}_{2k-1}$.

Remark 8. The Hamiltonian of the diagonal subgroup in SL_2 has the form

$$f=\sum_{i=0}^{k}(2k+1-2i)(-1)^i a_i a_{2k+1-i}.$$

The commutator of f and h is $\{h, f\} = 2h$. On the subspace of the original polynomial space formed by polynomials with the root $x = 0$ of multiplicity $\geq k + 2$ we have $f = h = 0$. Therefore $f = h = 0$ for any polynomial with a root of multiplicity $\geq k + 2$.

Proposition 6 is easily derived from the identity $h = 0$. Other identities for the coefficients of polynomials with multiple roots are given in [17, 18].

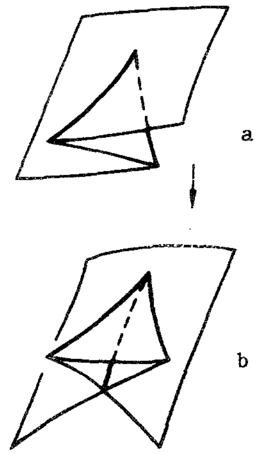


Fig. 1

Examples. For k = 2, the open swallowtail is a semicubic parabola on a plane. For k = 3, it is the two-dimensional cusp edge of the discriminant in the space of the polynomials $x^5+a_2x^3+a_3x^2+a_4x+a_5$, schematically shown in Fig. 1a. Differentiation of polynomials maps it on a swallowtail (Fig. 1b) — the discriminant in the space of polynomials of degree 4. Under this projection from $\mathbf{R}^4$ to $\mathbf{R}^3$, the open swallowtail acquires a self-intersection line. Hence its name.

5.2. Let us identify the polynomial space $\mathcal{P}_{2k-1}$ with the space of hyperelliptic curves of genus g = k − 1,

$$y^2=x^{2g+1}+\lambda_1x^{2g-1}+\ldots+\lambda_{2g},\ \lambda\in\mathbf{C}^{2g}.$$

The point λ outside the discriminant $\Delta \subset \mathbf{C}^{2g}$ corresponds to a hyperelliptic curve $V_\lambda \subset \mathbf{C}^2$ of genus g with one point at infinity. The homotopic type of the curve V_λ is a bouquet of 2g circles. Plane Gauss–Manin connectivity is defined in the fibration of the cohomology spaces $H^1(V_\lambda, \mathbf{C})$ over $\mathbf{C}^{2g}\backslash\Delta$: the basis of integer cycles on V_λ can be carried over by continuity to neighboring hyperelliptic curves. The period map [16] associates to the point $\lambda \in (\mathbf{C}^{2g}\backslash\Delta)$ a characteristic class of the area form $dx \wedge dy$ on the Lagrangian curve V_λ. We can locally consider the period map as a holomorphic map of the neighborhood of the point $\lambda \in (\mathbf{C}^{2g}\backslash\Delta)$ to the 2g-dimensional space $H^1(V_\lambda, \mathbf{C})$. It is shown in [36] that the Jacobian of this map does not vanish.

A nondegenerate skew-symmetric 2-form dual to the intersection index of the cycles on V_λ is defined in the space $H^1(V_\lambda, \mathbf{C})$. Applying the period map, we carry it over to the base of $\mathbf{C}_{2g}\backslash\Delta$. It is shown in [16] that this produces a symplectic structure on $\mathbf{C}_{2g}\backslash\Delta$ which is holomorphically continued to a symplectic structure on the entire space $\mathbf{C}_{2g}$. We call it the intersection form in the space of hyperelliptic curves.

Denote by $\Sigma \subset \Delta$ the closure of the set of points λ corresponding to curves V_λ with g double points.

Proposition 7. Σ is a Lagrangian manifold in the symplectic space of hyperelliptic curves.

The proof is given in [16].

The main reason why Σ is Lagrangian is that the g cycles of the hyperelliptic curve V_λ that vanish at the double points for $\lambda \to \lambda_0 \in \Sigma$ are pairwise nonintersecting and generate a Lagrangian subspace in $H^1(V_\lambda, \mathbf{C})$.

Examples. For g = 1, Σ is a semicubic parabola on the plane of elliptic curves. For g = 2, Σ is a surface in the space of the polynomials $x^5+\lambda_1x^3+\lambda_2x^2+\lambda_3x+\lambda_4$ formed by the self-intersection points of the discriminant in this space. Differentiation of polynomials maps Σ on the Maxwell stratum of the Lagrangian map A_4 (in the notation of [9]). The relative position of the Maxwell stratum and the caustic (= the discriminant of polynomials of degree 4) is shown in Fig. 2. We see that the Maxwell stratum is also a swallowtail. This is not accidental.

THEOREM 7 [16]. The Lagrangian manifold Σ in the symplectic space of hyperelliptic curves of genus g is isomorphic to the open swallowtail in the symplectic polynomial space $\mathcal{P}_{2g+1}$.

Proof. To the polynomial $P_n(x) = x^n + \lambda_1 x^{n-1} + \ldots$ of degree n = 2g + 1 associate the polynomial

$$Q_{2g+1}(x)=\mathrm{Res}_{t=\infty}[(t-x)^{2g-1}P_{2g+1}(t)]^{1/2}dt.$$

The expression under Res is the series in powers of t^{-1} of one of the branches of the function

$$t^{2g}\sqrt{\left(1-\frac{x}{t}\right)^{2g-1}\left(1+\frac{\lambda_1}{t}+\ldots\right)},$$

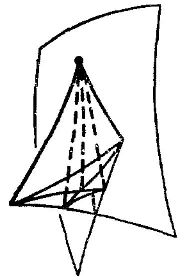


Fig. 2

convergent for fixed (x, λ) for sufficiently large $|t|$. The correspondence $P_{2g+1} \mapsto Q_{2g+1}$ is a quasihomogeneous map to the space of polynomials of degree $\leq 2g + 1$. Computing Q modulo $(\lambda)^2$, we obtain

$$Q(x) \equiv \mathrm{Res}_{t=\infty}(dt)\left[1 - \frac{2g-1}{2}\frac{x}{t} + \frac{2g-1}{2}\frac{2g-3}{2}\left(\frac{x}{t}\right)^2 + \dots\right] \times \\ \times \left[t^{2g} + \lambda_1 \frac{t^{2g-1}}{2} + \dots + \lambda_{2g-1}\frac{t^{-1}}{2}\right].$$

Since all the coefficients of the first factor are nonzero, the highest-order coefficient in Q (it is independent of λ) is nonzero and our map is a quasihomogeneous isomorphism of the spaces of polynomials with a fixed highest-order coefficient. In particular, if the sum of the roots of P is zero, then the sum of the roots of Q is also zero.

Let $P_{2g+1}(x) = (x - a)P_g^2(x)$. Then

$$Q_{2g+1}(a) = \mathrm{Res}_{t=\infty}(t-a)^g P_g(t)\, dt = 0.$$

We similarly verify that $Q'(a) = Q''(a) = \dots = Q^{(g)}(a) = 0$. Thus, if $P \in \Sigma$, then Q is a point of an open swallowtail. The isomorphism assertion now follows from irreducibility of the open swallowtail. The latter in turn follows from the fact that an open swallowtail is the image of the space of the polynomials $Q_g(x) = x^g + a_1x^{g-1} + \dots$ under the mapping $Q_g \mapsto (x - a_1/(g+1))^{g+1}Q_g$. (In fact, this map is a normalization of the open swallowtail.) It can be shown (see Corollary 1 of Theorem 11 in Sec. 7) that the quasihomogeneous symplectic structure in which Σ is Lagrangian is unique up to a multiplier. Therefore, the quasihomogeneous automorphism of the space $\mathscr{P}_{2g+1}$ constructed above also takes into one another two symplectic structures in this space.

Remark 9. a) The quasihomogeneous automorphism of polynomial spaces induces a transformation of a part of the coordinates having a lower quasihomogeneous degree. In particular, this induced transformation sends the Maxwell stratum of Fig. 2 to a caustic.

b) Applying Proposition 2, we can show (Corollary 2 of Theorem 10) that in both the complex and the real cases the germs of symplectic structures where an open swallowtail is Lagrangian are equivalent.

c) Generalizing Proposition 6, we will list the ranks of the restriction of the intersection form to the discriminant strata. Let the stratum points correspond to the curves V_λ with normalization of genus $p < g$. Then the sought rank is 2p (A. N. Varchenko, compare [16, Sec. 6]). In particular, Σ is characterized as the set of $\lambda \in C^{2g}$ for which the curve V_λ is rational. The assertion about the rank implies, in particular, that for $g > 1$ two symplectic structures in $\mathscr{P}_{2g+1}$ are different: curves of genus $p = [g/2]$ correspond to the points of the open swallowtail.

d) The intersection form defines the field of characteristic directions on the discriminant. At the point of the discriminant corresponding to the polynomial $(x - a)^2P_{2g-1}(x)$, the tangent space to the discriminant consists of all polynomials of degree $\leq 2g - 1$ for which a is a root and the characteristic direction is generated by the vector $(x - a)P_{2g-1}'(x)$ in the tangent space. Omitting the details, we will only stress the main point of the proof: the form $(x - a)P_{2g-1}'(x)(dx)/y$ is a total differential on the normalization $t^2 = P_{2g-1}(x)$, $t = y/(x - a)$, of the curve $y^2 = (x - a)^2P_{2g-1}(x)$.

e) The intersection form is constructed in [16] in a much more general setting. We can start with an arbitrary irreducible germ of a plane curve at a singular point (instead of starting with $y^2 = x^{2g+1}$), take as the family V_λ its R-miniversal deformation, and construct the period map using the germ of the area form in general position holomorphically dependent on λ. Under these conditions, the base of the deformation is of even dimension $\mu = \dim H^1(V_\lambda, C)$, and the intersection index of the cycles

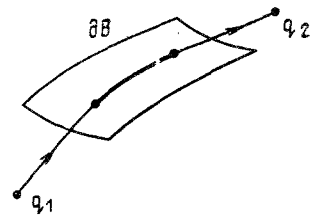


Fig. 3

on V_λ is nondegenerate. It is proved in [16] that the resulting intersection form on the base of the deformation is holomorphically continued on the discriminant to the germ of a symplectic structure in the entire parameter space and is independent of the choice of the area form up to a discriminant-preserving diffeomorphism.

The manifold Σ of the curves V_λ with $\mu/2$ double points is nonempty and Lagrangian with respect to the intersection form. Thus, the series of open swallowtails is embedded, by Theorem 6, in the wider class of singular Lagrangian manifolds. We will see that open swallowtails play an important role in the problem of going past an obstacle and in other applications. However, the role of their generalizations, and equally the role of the intersection form in the space of hyperelliptic curves, is not clear: no interesting applications have been found so far.

6. SINGULARITIES OF RAY SYSTEMS IN THE PROBLEM OF GOING PAST AN OBSTACLE

The problem of going past an obstacle requires finding the extremals of the length functional on a Riemannian manifold with an edge. An edge is a nonsingular hypersurface ∂B — the boundary of the obstacle — on the manifold B without an edge. The extremal between the points $q_1, q_2 \in B$ in the problem of going past an obstacle consists of segments of geodesics in B tangent to ∂B that originate from the points q_1, q_2 and a segment of a geodesic on ∂B connecting the points and the directions of tangency (Fig. 3). Treating q_2 as a variable point, we obtain a bundle of extremals that separate from some bundle γ of geodesics on ∂B in all directions tangent to γ.

Fix the bundle γ of geodesics on ∂B, i.e., the set of oriented geodesics leaving some nonsingular initial hypersurface on ∂B in the direction of its positive normal.

Definition. The ray system of the bundle γ is the set of oriented geodesics on B tangent to the geodesics of the bundle γ.

The Riemannian metric on B defines the Euclidean structure $\langle\, , \rangle$ in the fibers of the fibration T^*B. The characteristics of the hypersurface $\langle p, p\rangle = 1$ in T^*B are projected to oriented geodesics on B. A canonical symplectic structure exists on the manifold of the characteristics of the hypersurface in the symplectic manifold (defined at least locally) (see [10, p. 52]). Therefore, oriented geodesics on B locally form a symplectic manifold. The ray system of the bundle γ in general position on ∂B is a Lagrangian manifold in the geodesic space. Rays asymptotic to ∂B are its singular points.

Example 1. The manifold of oriented geodesics in the Euclidean space $B = R^n$ is symplectomorphic to a cotangent fibration of the sphere T^*S^{n-1} up to the sign of the symplectic structure (see [4] and [10, p. 53]). Let $n = 2$ and let ∂B be a curve in general position on the plane. Then the ray system is the curve dual to ∂B. Rays tangent to ∂B at inflection points are the cusp points of the dual curve, the rays bitangent to ∂B are its double points.

Definition. The triad (L, l, H) in the symplectic manifold M consists of a nonsingular Lagrangian manifold L, a nonsingular hypersurface $l \subset L$ (l is an isotropic manifold), and a hypersurface $H \subset M$ tangent to L at the points *of* l. Equivalent triads are moved term by term to one another by the symplectomorphism M.

A germ of the triad (L, l, H) corresponds to a germ of the Lagrangian manifold Λ in the space of characteristics of the hypersurface H formed by the characteristics through l. The characteristics tangent to l are the singular points of Λ.

Remark 10. At the quadratic tangency points $x \in l$ of H to L, the tangent space to L is projected along the characteristics of the hypersurface H on a Lagrangian space. At the points of Λ, we have the field $x \mapsto T(x)$ of Lagrangian spaces that analytically continues the field of tangent spaces to Λ to its singular points. Continuous deformation of the triad produces a continuous variation of the field T. The map $\pi: l \to M'$ whose image is the nested Lagrangian manifold with the field $x(\in l) \mapsto T(x)$ of Lagrangian spaces tangent to $\Lambda = \pi(l)$ at nonsingular points $(T(x) = T_{\pi(x)}\Lambda)$ can be completed to a triad in M.

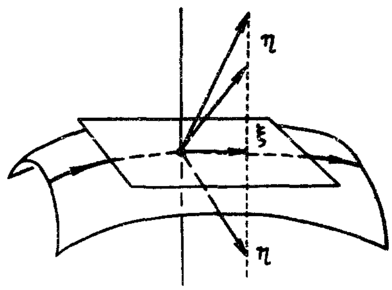


Fig. 4

Example 2. With the bundle γ of geodesics on ∂B associate a triad in T^*B. Let $H = \{(p, q) \mid \langle p, p\rangle = 1\}$ be the manifold of all unit vectors, let L consist of all possible continuations of the velocity covectors ξ of the geodesics of the bundle γ to covectors η on B (Fig. 4), and let $l = H \cap L$ be the manifold of velocity unit vectors of the geodesics from γ. It is easy to verify that (L, l, H) is a triad, and H is strictly quadratically tangent to L (Fig. 4) because of the convexity of the scalar square $\langle\eta, \eta\rangle$, The Lagrangian manifold Λ of this triad is the ray system of the bundle γ.

Example 3. In the notation of 5.1, we define a triad τ_n in the space of polynomials of the form

$$\frac{x^{2n}}{(2n)!}+a_1\frac{x^{2n-1}}{(2n-1)!}+\ldots+\frac{a_{2n}}{0!},$$

setting $H=h^{-1}(0)$, $L=\{a \mid a_{n+1}=\ldots=a_{2n}=0\}$, $l=H\cap L$.. Clearly, H is strictly quadratically tangent to L along l. Denote by $\tau_{n+k,k}$ the suspension $(L\times\mathbf{R}^k,\ l\times\mathbf{R}^k,\ H\times T^*\mathbf{R}^k)$ of the triad τ_n, $k = 0,1,2,\ldots$. The Lagrangian manifold Λ of the triad $\tau_{n+k,k}$ is the product of $\mathbf{R}^k$ and an open swallowtail in the polynomial space $\mathscr{P}_{2n-1}$.

THEOREM 8 [19]. The triads $\tau_{m,k}$, $m \geq k$, are stable. The germ of a triad in general position at the quadratic tangency point of a hypersurface to a Lagrangian manifold is equivalent to the germ at zero of one of the triads $\tau_{m,k}$.

Proof. The equation $h=a_{2n}-a_{2n-1}a_1+\ldots\pm a_n^2/2=0$ of the hypersurface of the triad τ_n is linear in $(a_{n+1},\ldots, a_{2n})$. For an arbitrary germ of the triad (L, l, H) in $\mathbf{R}^{2m}$, we locally identify $\mathbf{R}^{2m}$ with T^*L so that the intersection of H with each fiber of the fibration $T^*L \to L$ is an affine hyperplane. To this end, we complete the equation of H to a system of Darboux coordinates. The projection along the coordinate Lagrangian space transversal to L defines in $\mathbf{R}^{2m}$ a Lagrangian fibration structure $T^*\mathbf{R}^m$ with the required property. Fiberwise translations by the vectors of the Lagrangian section $L \subset T^*\mathbf{R}^m$ identify $T^*\mathbf{R}^m$ with T^*L without destroying the linearity of the intersections with the fibers of the fibration.

The dual object to the field of affine hyperplanes in cotangent spaces to L is the field v of tangent straight lines on L together with the field ξ of 1-forms on these lines. The field v extends to the entire L the field of characteristic directions of the hypersurface H defined at the points of $l \subset L$ and tangent to L at these points. For a triad in general position, the projection of l along the integral curves of the field v has only Whitney singularities, i.e., there exist coordinates $x, \lambda_1, \ldots, \lambda_{n-1}$ on L in which $v = \langle\partial/\partial x\rangle$ and l is defined by the equation

$$G=x^n+\lambda_1x^{n-2}+\ldots+\lambda_{n-1}=0,\ n\leqslant m.$$

Since H is quadratically tangent to L, we have $\xi = \varphi(x, \lambda)G^2dx$, where $\varphi(0, 0) \neq 0$.

The family of semiforms $G(\varphi dx)^{1/2}$ on a straight line fibered over λ is reduced to the form $y=y(x,\lambda)$, $\Lambda=\Lambda(\lambda)$ by the substitution of variables $y = y(x, \lambda)$, $\Lambda = \Lambda(\lambda)$.

This follows from the versality theorem for the semiforms $f(x)(dx)^{1/2}$ relative to the group of orientation-preserving diffeomorphisms of the straight line (see [19]). The versality theorem for the degrees of volume forms is discussed in Appendix 1.

We have thus normalized the hypersurface $H \subset T^*L$. It has the required form in the coordinates $a_n=G(y,\Lambda)$, $a_{n-1}=\partial G/\partial y$, $a_{n-2}=\partial^2G/\partial q^2, \ldots$ on L and the dual coordinates $a_{n+1}, a_{n+2}, \ldots$ on L^*.

COROLLARY. In the problem of going past an obstacle in general position, the germ of the ray system corresponding to the bundle of geodesics in general position on the boundary of the obstacle is symplectically diffeomorphic to the germ of an open swallowtail or the germ of the union of two nonsingular Lagrangian manifolds at a transversal intersection point.

In order to prove the corollary, it suffices to verify that for the edge ∂B and the bundle of geodesics in general position on the edge, the triad of Example 2 satisfies the condition of general position assumed in the proof of Theorem 7. The last condition is a condition on the location of the manifold of velocity unit vectors of the bundle of geodesics on ∂B in relation to the geodesic flow on the manifold of unit vectors on B. The bundle of geodesics on ∂B is defined by an arbitrary nonsingular Lagrangian submanifold in the space of geodesics on ∂B. Omitting trivial details, we note the following proposition, which is useful for proving the corollary.

On the set Φ of unit vectors in B applied at the points of ∂B there is a stratification by orders of tangency to ∂B: $\Phi \supset \Phi_0 \supset \Phi_1 \supset \Phi_2 \supset \ldots$ (tangent, asymptotic, biasymptotic, etc., unit vectors). It is shown in [4] that for an edge ∂B in general position the projections of Φ to the space of rays on B along the characteristics on the manifold H of all the unit vectors have only Whitney singularities — a fold at the points of $\Phi_0 \backslash \Phi_1$, a pleat at the points of $\Phi_1 \backslash \Phi_0$, etc. On the other hand, a geodesic flow of the edge ∂B is defined on the manifold Φ_0 of the unit vectors tangent to ∂B. For an edge in general position, the projection of $\Phi_1 \subset \Phi_0$ to the space of geodesics on ∂B also has only Whitney singularities. The second stratification Φ_1 arising in this case coincides with the first stratification.

Proposition 8. For a hypersurface $\partial B \subset B$ in general position, the projection of the set Φ_1 of asymptotic unit vectors to the space of geodesics on ∂B has a fold at the points of $\Phi_2 \backslash \Phi_3$, a pleat at the points of $\Phi_3 \backslash \Phi_4$, etc.

This proposition is a multidimensional generalization of the following theorem of classical differential geometry of surfaces in $\mathbf{R}^3$: the asymptotic direction at the inflection point of an asymptotic line on a surface of negative curvature is biasymptotic, and conversely.

7. PROPERTIES OF OPEN SWALLOWTAILS

7.1. Denote by $\Sigma_k(n)$ the manifold of polynomials of the form

$$F_n(x, a) = \frac{x^n}{n!} + a_2 \frac{x^{n-2}}{(n-2)!} + \ldots + a_n,$$

having a root of multiplicity $\geq n - k$. For $k \leq n - 2$, $\Sigma_k(n)$ is a k-dimensional singular manifold, $\Sigma_{n-2}(n)$ is the discriminant in the polynomial space $\mathcal{P}_n$, $\Sigma_k(2k + 1)$ is the open swallowtail in the space $\mathcal{P}_{2k+1}$.

Differentiation of polynomials defines a map of degree 1, $\Sigma_k(n + 1) \to \Sigma_k(n)$, of manifolds of equal dimension. We obtain a tower of maps

$$\Sigma_k(k+2) \leftarrow \Sigma_k(k+3) \leftarrow \ldots \leftarrow \Sigma_k(n) \leftarrow \ldots . \tag{1}$$

General Noetherian considerations suggest that this tower is stabilized on the final story.

THEOREM 9 (V. I. Arnol'd [18]). The map $\Sigma_k(n + 1) \to \Sigma_k(n)$ is an isomorphism for $n \geq 2k + 1$, i.e., the tower (1) is stabilized starting with the open swallowtail.

The theorem is easily proved using the identities $h = f = 0$ from 5.1 for the coefficients of polynomials with multiple roots.

7.2. Let us describe the algebra of regular functions on a stable manifold. Its normalization Σ_k is nonsingular. The coordinates on the normalization are $(x, a_2, \ldots, a_k)$, where x is a root of the polynomial $F_{2k+1}(\cdot, a)$ of multiplicity $\geq k+1$.

THEOREM 10. 1) The subalgebra $\mathcal{O}_{\Sigma_k(2k+1)} \subset \mathcal{O}_{\Sigma_k}$ coincides with the space of functions of the form

$$\int_0^x Q(\xi, a) F_k(\xi, a)\, d\xi + C(a),$$

where $Q \in \mathcal{O}_{\Sigma_k}$, $C \in \mathcal{O}_{\mathcal{P}_k}$.

2) $\mathcal{O}_{\Sigma_k(2k+1)}$ is a free module over $\mathcal{O}_{\mathcal{P}_{k+1}}$ with generators $1, a_{k+2}, \ldots, a_{2k+1}$.

Proof. 1°. By induction on i we easily verify

LEMMA [18]. In $\mathcal{O}_{\Sigma_k}$ we have the equality

$$a_{k+i} = \frac{(-1)^i}{i!} \int_0^x F_k(\xi, a)\, d\xi^i, \quad i = 1, 2, 3, \ldots .$$

2°. The miniversal deformation of the semiform $x^{k+1}(dx)^{1/2}/(k+1)!$ is given by (see Sec. 6) $F_{k+1}(x, a)(dx)^{1/2}$. The condition of infinitesimal versality implies that any function $\varphi(x, a_2, \ldots, a_{k+1})$ is representable as

$$\varphi(x, a) = F_k(x, a) R(x, a) + \frac{1}{2} F_{k+1}(x, a) R'(x, a) + \sum_{i=1}^{k} \lambda_i(a) \frac{x^{i-1}}{(i-1)!} \tag{2}$$

Multiplying (2) by F_{k+1}, we obtain

$$\varphi F_{k+1} = \frac{\partial}{\partial x}(RF_{k+1}^2/2) + \sum_{i=1}^{k} \lambda_i \frac{x^{i-1}}{(i-1)!} F_{k+1}. \tag{3}$$

If $F_{2k+1} \in \Sigma_k(2k+1)$ and x is a root of F_{2k+1} of multiplicity $\geq k+1$, then x is a root of F_{k+1}. Integrating (3), we find

$$\int_0^x \varphi F_{k+1} d\xi = \sum_{i=1}^{k} \lambda_i(a) \int_0^x \frac{\xi^{i-1}}{(i-1)!} F_{k+1}(\xi, a)\, d\xi - R(0, a) F_{k+1}^2(0, a)/2.$$

Integrating by parts, we obtain that the function $\int_0^x Q(\xi, a) F_k(\xi, a)\, d\xi + C(a)$ is representable as

$$\sum_{i=1}^{k} \lambda_i(a_2, \ldots, a_{k+1}) a_{k+i+1} + \lambda_0(a_2, \ldots, a_{k+1}). \tag{4}$$

In particular, the space of these functions lies in $\mathcal{O}_{\Sigma_k(2k+1)}$.

3°. Let $A(x, a) \in \mathcal{O}_{\Sigma_k(2k+1)}$. Then $A = B(a_2, \ldots, a_{2k+1})$, and

$$\frac{\partial A}{\partial x} = \sum_{i \geq 1} \frac{\partial B}{\partial a_{k+i}} \frac{\partial a_{k+i}}{\partial x} = F_k(x, a) \left[\sum_{i \geq 1} \frac{\partial B}{\partial a_{k+i}} \frac{(-1)^i}{(i-1)!} x^{i-1} \right],$$

whence follows the converse inclusion. This completes the proof of part 1.

4°. k-fold differentiation of polynomials defines the finite map $\Sigma_k(2k+1) \to \mathcal{P}_{k+1}$ of degree k + 1. Hence it follows that the rank of the $\mathcal{O}_{\mathcal{P}_{k+1}}$-module $\mathcal{O}_{\Sigma_k(2k+1)}$ is k + 1. The representation (4) implies that $1, a_{k+2}, \ldots, a_{2k+1}$ are the generators of this module.

COROLLARY. The projection $\Sigma_k(2k+1) \to \mathcal{P}_{k+1}$ is miniversal.

Proof. Let $\deg a_i = i$ be a quasihomogeneous gradation in $\mathcal{O}_{\Sigma_k(2k+1)}$. Then $\deg a_i a_j > 2k+1$ for $i, j \geq k+2$. Therefore $\mathfrak{m}^2_{\Sigma_k(2k+1)} \subset (a_2, \ldots, a_{k+1})$. By Remark 4, this implies that versality is independent of the symplectic structure and is determined by the projection $\Sigma_k(2k+1) \to \mathcal{P}_{k+1}$. By part 2 of the theorem, it follows that $(1, a_{k+2}, \ldots, a_{2k+1})$ is a basis in the local algebra.

7.3. We are describing vector fields tangent to an open swallowtail using an axiomatic definition of a stable manifold.

Consider the germ π: (X, 0) → (Y, 0) of a finite analytic map of nonsingular manifolds of equal dimension in which singularity of codimension 1 is in general position, i.e., it is a fold.

Let $S \subset X$ be the hypersurface of fold points, $\bar{S}$ its closure in X, s = 0 a simple equation of $\bar{S}$. The field of directions v on X is defined at the points from S: this is the field of kernels of the differential of the map π. The map π defines the embedding $\pi^*: \mathcal{O}_Y \hookrightarrow \mathcal{O}_X$. Denote by $\mathcal{A}$ the space $\mathcal{O}_Y \hookrightarrow \mathcal{A} \hookrightarrow \mathcal{O}_X$ of germs of functions whose derivative in the direction of the field v vanishes at the points of S. It is easy to show that $\mathcal{A}$ is a finitely generated closed subalgebra in $\mathcal{O}_X$. Therefore $\mathcal{A} = \mathcal{O}_Z$ for some germ of the analytic set Z, X → Z → Y.

Definition. Z is called a universal manifold of the map π.

Example. Part 1 of Theorem 10 implies that the open swallowtail $\Sigma_k(2k+1)$ is a universal manifold of the Whitney map $\pi: \Sigma_k \to \mathcal{P}_{k+1}$.

Now let $X \hookrightarrow T^*Y \to Y$ be a germ of a Lagrangian manifold in general position, Φ its graph. Then $\mathcal{O}_\Phi \subset \mathcal{O}_Z$, because at the fold points of the projection π: X → Y the field v lies in the kernel of the map X → Φ.

Proposition 9. The diffeomorphism of the graph Φ is liftable to a diffeomorphism of the universal manifold Z → Φ.

Proof. The diffeomorphism g of the graph $\Phi \subset J^0Y$ is liftable to the diffeomorphism $\tilde{g}$ of the Legendrian manifold $X \hookrightarrow J^1Y$ of its contact elements. The field v is invariantly defined by the projection X → Φ. Therefore the subalgebra $\mathcal{O}_Z \subset \mathcal{O}_X$ is invariant relative to $\tilde{g}^*$.

COROLLARY. The vector field in the space $\mathscr{P}_{k+2}$ tangent to the discriminant $\Sigma_k(k + 2)$ is liftable to a vector field in the space $\mathscr{P}_{2k+1}$ tangent to the open swallowtail $\Sigma_k(2k + 1)$.

Indeed, $\Sigma_k(k + 2)$ is the graph of a stable Lagrangian map $\Sigma_k \hookrightarrow T^*\mathscr{P}_{k+1} \to \mathscr{P}_{k+1}$ of type A_k (in the notation of [9]).

THEOREM 11 [5, 18]. The vector field in $\mathscr{P}_n$, $k+2 \leqslant n \leqslant 2k+1$ tangent to the discriminant $\Sigma_{n-2}(n)$ is liftable in $\Sigma_{n-2}(n)$ to a vector field tangent to the open swallowtail $\Sigma_k(2k + 1)$. Conversely, each vector field tangent to the open swallowtail $\Sigma_k(2k + 1)$ is representable as the sum of liftings of fields tangent to discriminants in the spaces of polynomials of degree n, $k + 2 \le n \le 2k + 1$.

Proof. The theorem follows from these assertions.

1°. Corollary of Proposition 9.

2°. The vector field tangent to $\Sigma_n(2n + 1)$ is tangent to $\Sigma_k(2n + 1) \subset \Sigma_n(2n + 1)$ for $n \ge k$.

3°. Theorem 9: $\Sigma_k(2n + 1) \to \Sigma_k(2k + 1)$ is an isomorphism.

4°. A vector field in $\mathscr{P}_{n+1}$ is representable as the sum of a field tangent to the discriminant and the lifting of a field from the space $\mathscr{P}_n$ (part 1 of Theorem 4 applied to a stable Lagrangian map of type A_n).

COROLLARY 1. The symplectic structure in $\mathscr{P}_{2k+1}$ in which the manifold $\Sigma_k(2k + 1)$ is Lagrangian is the sum of the quasihomogeneous symplectic structure ω from 5.1 and a 2-form of higher quasihomogeneous degree. In particular, the quasihomogeneous symplectic structure in which $\Sigma_k(2k + 1)$ is Lagrangian is unique up to a numerical multiplier.

Proof. By the lemma of Sec. 1, the symplectic structure in which $\Sigma_k(2k + 1)$ is Lagrangian is representable in the form $di_V\omega$, where V is a vector field tangent to $\Sigma_k(2k + 1)$. It therefore suffices to show that the quasihomogeneous field V is of degree 0 only if it is proportional to an Euler field.

The basis of the module of fields tangent to the discriminant in the polynomial space is described by part 2 of Theorem 4. It is easy to see that the Euler field is the only one among them with zero quasihomogeneous degree, while all the others are of positive degree. The components of the vertical vector field in $\mathscr{P}_{2k+1}$ tangent to $\Sigma_k(2k + 1)$, i.e., the field which is the lifting of the zero field in $\mathscr{P}_n$ for $n \le 2k$, vanish on $\Sigma_k(2k + 1)$ because the projection $\Sigma_k(2k+1) \to \mathscr{P}_n$ is an immersion at nonsingular points. By part 2 of Theorem 10, the degree of a function that vanishes on $\Sigma_k(2k + 1)$ is not less than 2 in the variables $(a_{k+2}, \dots, a_{2k+1})$, and its quasihomogeneous degree is thus $\ge 2k + 4$. Therefore, the quasihomogeneous degree of the vertical field tangent to $\Sigma_k(2k + 1)$ is positive (≥ 3). The corollary now follows from Theorem 11.

COROLLARY 2. The germs of symplectic structures in which the product of an open swallowtail and a singular manifold is Lagrangian are equivalent.

Proof. By Proposition 2, the corollary follows from these facts: the manifold $\Sigma_k(2k + 1)$ is homogeneous and reduced, any germ of a symplectic structure in which $\Sigma_k(2k + 1)$ is Lagrangian is homotopic to one of the structures $\pm\omega$ in the class of these structures (Corollary 1), and the structures $\omega \sim -\omega$ (defined by the automorphism $a_i \mapsto (-1)^i a_i$ of the manifold $\Sigma_k(2k + 1)$) are equivalent.

7.4. THEOREM 12 (V. I. Arnol'd). The germ at zero of the diagram $\Sigma_k(2k+1) \hookrightarrow X^{2k} \to Y$, where $X^{2k} \to Y$ is m-dimensional fibration in general position, $m \le k$, is reducible to the normal form $\Sigma_k(2k+1) \hookrightarrow \mathscr{P}_{2k+1} \to \mathscr{P}_{2k+1-m}$, where the map of the polynomial space is defined by m-fold differentiation.

The proof is by induction on m.

Represent the fibration $X^{2k} \to Y$ as the composition $X^{2k} \to Z \to Y$ of a $(m - 1)$-dimensional fibration and a 1-dimensional fibration in general position in relation to $\Sigma_k(2k + 1)$. By the inductive hypothesis, the diagram $\Sigma_k(2k+1) \hookrightarrow X^{2k} \to Z$ is equivalent to the standard diagram $\Sigma_k(2k+1) \hookrightarrow \mathscr{P}_{2k+1} \to \mathscr{P}_{2k+2-m}$. The fiber of the one-dimensional fibration $\mathscr{P}_{2k+2-m} \to Y$ in general position is transversal to the discriminant $\Sigma_{2k-m}(2k + 2 - m)$ in the polynomial space. By the corollary of Theorem 6 applied to a Lagrangian map of type A_{2k+1-m}, this one-dimensional fibration is reduced to standard form by a discriminant-preserving diffeomorphism which is homotopic to the identity diffeomorphism in the class of these diffeomorphisms. By Theorem 11, this diffeomorphism is liftable in $\mathscr{P}_{2k+1}$ to a diffeomorphism preserving $\Sigma_k(k + 1)$. It reduces the diagram $\Sigma_k(2k+1) \hookrightarrow X \to Y$ to the required normal form.

COROLLARY 1. The image of the projection $\Sigma_k(2k+1) \hookrightarrow X^{2k} \to Y^n$ in general position is locally diffeomorphic to $\Sigma_k(n + 1)$ for $n > k$.

COROLLARY 2. The germ of the diagram $\Sigma_k(2k+1) \hookrightarrow X^{2k} \to Y^k$ in general position is reducible to normal form by a diffeomorphism of the base and by a fiberwise affine transformation of the fibration $X^{2k} \to Y_k$.

Proof. By Theorem 12 and corollary of Theorem 10, the projection $\Sigma_k(2k + 1) \to Y^k$ is miniversal. Therefore Proposition 3 implies that any identity diffeomorphism over the base of the diagram $\Sigma_k(2k+1) \hookrightarrow X^{2k} \to Y^k$ restricted to $\Sigma_k(2k + 1)$ can be replaced with a fiberwise affine transformation.

Proposition 10. Let $\Lambda \hookrightarrow T^*B \to B$ be a germ of the Lagrangian map of the manifold Λ diffeomorphic to the product of the open swallowtail $\Sigma_k(2k + 1)$ and a nonsingular n-dimensional manifold. Assume that the fiber $T_0{}^*B$ of the Lagrangian fibration is transversal to the tangent Lagrangian space of the manifold Λ at the origin. Then the germ of the Lagrangian map is equivalent to the suspension of the standard Lagrangian map $\Sigma_k(2k+1) \hookrightarrow (\mathcal{P}_{2k+1}, \omega) \to \mathcal{P}_{k+1}$ defined by k-fold differentiation of polynomials.

Proof. Let $\Lambda = \Sigma_k(2k + 1) \times D^n$. Under the conditions of the proposition, the projection of $\{0\} \times D^n$ to B is nonsingular. Restrict the Lagrangian map to the transversal $Y^k \subset B$ to the image of this projection at the origin. $T_Y{}^*B \cap \Lambda$ is diffeomorphic to $\Sigma_k(2k + 1)$. The fiber of the projection $T_Y{}^*B \to T^*Y$ is transversal, by assumption, to the 2k-dimensional tangent space to the manifold $T_Y{}^*B \cap \Lambda \subset T_Y{}^*B$ at the origin. Therefore the restriction to Y^k is a Lagrangian map in general position of the manifold $\Sigma_k(2k + 1)$. If we prove that it is equivalent to the standard map, then by Proposition 4 this would imply that the original Lagrange map is equivalent to the suspension of the standard map.

The diagram $\Sigma_k(2k+1) \hookrightarrow T^*Y^k \to Y^k$, by Theorem 12, is reducible to standard normal form by a fibered diffeomorphism. This normal form is quasihomogeneous. Therefore this diffeomorphisms sends a symplectic structure to a structure homotopic to $\pm\omega$ in the class of zero structures on $\Sigma_k(2k + 1)$ and on the fibers of the fibration (Corollary 1 of Theorem 11). For any such symplectic structure, a Lagrangian map is versal (corollary of Theorem 10). It now follows from Theorem 3 that our Lagrangian map is equivalent to a standard map, because the structures ω and $-\omega$ define equivalent Lagrangian maps.

8. SINGULARITIES OF MULTIVALUED SOLUTIONS OF FIRST-ORDER SYSTEMS OF EQUATIONS

A system of first-order partial differential equations on the manifold Y is defined by the submanifold H in the space $J^1(y, \mathbf{R}^m)$ of 1-jets of sections of the fibration $\mathbf{R}^m \times Y \to Y$, where m is the number of sought functions. The section U of this fibration is called a solution of the system of equations if its 1-graph

$$j^1U = \{(u, p, q) \mid q \in Y,\ u = U(q),\ p_i = d_q U_i,\ i = 1, \ldots, m\}$$

is contained in H.

The 1-graph of a section is an n-dimensional integral manifold of the Cartan distribution on $J^1(Y, \mathbf{R}^m)$ defined by the equations $du_i = \sum_{j=1}^{n} p_i^j dq_j,\ i = 1, \ldots, m.$

The projections $J^1(Y, \mathbf{R}^m) \to J^0(Y, \mathbf{R}^m) \overset{\triangle}{=} \mathbf{R}^m \times Y^n \to Y^n$ are defined.

Definition. A multivalued section of the fibration $\mathbf{R}^m \times Y^n \to Y^n$ is a n-dimensional nonsingular integral manifold of the Cartan distribution whose projection to Y is almost everywhere a submersion. The image of the projection of a multivalued section in $J^0(Y, \mathbf{R}^m)$ is called its graph. Multivalued sections are called equivalent if their graphs are mapped into one another by the automorphism of the affine fibration $\mathbf{R}^m \times Y \to Y$.

Example. For m = 1, the Cartan distribution is a contact structure in J^1Y, the multivalued section is a nonsingular Legendrian submanifold. The graph of the multivalued section $X \hookrightarrow J^1Y$ is the graph of the Lagrangian map $X \hookrightarrow T^*Y \to Y$, where embedding is defined by the projection $J^1Y \to T^*Y : (u, p, q) \mapsto (p, q)$. Equivalence of Lagrangian maps is equivalence of the corresponding multivalued sections.

In this section we present the results of V. V. Lychagin on the classification of singularities of multivalued sections of fibrations of dimension m > 1 on the assumption that the projection $X \to Y$ has a Whitney singularity. For m = 1, this classification was obtained in [2, 9]: singularities in general position are stable and are reducible to A_μ normal forms, while their graphs are locally diffeomorphic to the discriminants $\Sigma_{\mu-1}(\mu + 1)$ in the polynomial spaces $\mathcal{P}_{\mu+1}$.

Let $X^n \hookrightarrow J^1(Y^n, \mathbf{R}^m)$ be a germ of a multivalued section in general position. Then a singularity of codimension 1 of the projection $X \to Y$ is a fold. Denote by Z the universal manifold of the map $X \to Y$ (for definition see 7.3), by $\Phi \subset \mathbf{R}^m \times Y$ the graph of the multivalued section X.

Proposition 11. The projection of a multivalued section on its graph is decomposable into a composition of maps $X \to Z \to \Phi$.

Proof. The subalgebra $\mathcal{O}_\Phi \subset \mathcal{O}_X$ is generated by the functions u_i and q_j. At the fold point, the derivative $L_v q_j$ along the vector v from the kernel of the projection $X \to Y$ vanishes by definition. The derivative $L_v u_i$ vanishes by the Cartan relationships $du_i = \sum p_{ij} dq_j$ and by regularity of the functions p_{ij} on X. Therefore $\mathcal{O}_\Phi \subset \mathcal{O}_Z$.

COROLLARY. If a germ of the projection $X^n \to Y^n$ of a multivalued section in general position of the fibration $\mathbf{R}^m \times Y^n \to Y^n$ has a Whitney singularity of degree n + 1, then its graph is diffeomorphic to the germ at zero of the manifold $\Sigma_n(m + n + 1)$.

Proof. The graph $\Phi \subset \mathbf{R}^m \times Y^n$ is the image of the universal manifold $\Sigma^n = \Sigma_n(2n + 1) \subset \mathbf{R}^n$ under the projection $\mathbf{R}^N \to \mathbf{R}^m \times Y^n$. The proposition will follow from Corollary 1 of Theorem 12 if we show that the fiber of the projection $\mathbf{R}^N \to \mathbf{R}^m \times Y^n$ is in general position to the manifold Σ^n at zero.

The manifold Σ^n is the graph of some multivalued section of the fibration $\mathbf{R}^N \to Y$. This follows from the fact that the field of tangent spaces to Σ^n is analytically continuable to singular points (see Remark 10 in Sec. 6). A small perturbation of the fibration $\mathbf{R}^n \to \mathbf{R}^m \times Y$ defines a small perturbation of this multivalued section, and with it a small perturbation of the original multivalued section X^n. The perturbed section also has a Whitney projection $X^n \to Y^n$ by stability of the Whitney map. Therefore for X^n in general position the diagram $\Sigma^n \hookrightarrow \mathbf{R}^n \to \mathbf{R}^m \times Y^n$ is also in general position.

A stable multivalued section $X^n \hookrightarrow J^1(Y^n, \mathbf{R}^m)$ is called simple if its germs at points from X belong to a finite number of equivalence classes.

The manifold $\Sigma_k(N) \hookrightarrow \mathscr{P}_N$ is the graph of a standard multivalued section of the fibration $\mathscr{P}_N \to \mathscr{P}_{k+1}$. Its suspension is defined by the product of the diagram $\Sigma_k(N) \hookrightarrow \mathscr{P}_N \to \mathscr{P}_{k+1}$ and a nonsingular manifold.

THEOREM 13 (V. V. Lychagin [26]). For $N \geq 2k + 1$ the germ at zero of a standard multivalued section is simple. Conversely, let $k \leq m$ and let $X^n \hookrightarrow J^1(Y^n, \mathbf{R}^m)$ be a multivalued section in general position. Then its germs at almost all the points where the projection $X^n \to Y^n$ has a Whitney singularity of degree k + 1 are equivalent to the suspension of a standard multivalued section.

Proof. Let $\Sigma^n \simeq \Sigma_k(2k+1) \times \mathbf{R}^{n-k}$ be a universal manifold of the Whitney projection $X^n \to Y^n$. The points where the (k + n)-dimensional tangent space $(\mathfrak{m}_{\Sigma^n}/\mathfrak{m}^2_{\Sigma^n})^*$ to Σ^n is meromorphically projected to $\mathbf{R}^m \times Y^n$ form a dense subset in $\{0\} \times \mathbf{R}^{n-k}$ if $k \leq m$. At such a point, the germ of the graph $\Phi \subset \mathbf{R}^m \times Y^n$ is diffeomorphic to a germ of Σ^n. The diagram $\Sigma^n \hookrightarrow \mathbf{R}^m \times Y^n \to Y^n$ in general position in the neighborhood of a general point from $\{0\} \times \mathbf{R}^{n-k}$ is reducible to standard normal form by a local automorphism of the affine fibration $\mathbf{R}^m \times Y^n \to Y^n$ by Corollary 2 of Theorem 12 (more precisely, by its (n − k)-parameter variant).

Remark 11. a) For $N \geq 2k + 1$, the graph of a standard multivalued section is an open swallowtail.

b) V. V. Lychagin also proved that for $m > 1$ there exist no other simple singularities of multivalued sections with a Whitney projection, except those listed in Theorem 13. In particular, for $1 < m < k$, there are no simple singularities.

Chapter 3: SIMPLE SINGULARITIES OF LAGRANGIAN MAPS

9. COXETER GROUPS AND LAGRANGIAN CURVES

A Coxeter group is a finite group W of linear transformations of the Euclidean space V^μ generated by reflections in hyperplanes. Each Coxeter group is the direct sum of the following irreducible groups (see [13], the index equals the dimension of V):

a) $I_2(p)$ $(p \geq 3)$ — the symmetry group of a regular p-gon;

b) A_μ $(\mu \geq 1)$ — the symmetry group of a simplex in R^μ $(A_2 \simeq I_2(3))$, D_μ $(\mu \geq 4)$, E_6, E_7, E_8;

c) B_μ $(\simeq C_\mu, \mu \geq 2)$ — the symmetry group of the μ-dimensional cube $(B_2 \simeq I_2(4))$, F_4;

d) H_2 $(\simeq I_2(5))$, H_3 — the symmetry group of the icosahedron, H_4;

e) G_2 $(\simeq I_2(6))$.

The space of orbits $V^{\mathbf{C}}/W$ is isomorphic to $\mathbf{C}^\mu$ by Chevalley's theorem (see [13]). The manifold $\Phi \subset \mathbf{C}^\mu$ of irregular orbits is called the discriminant of the Coxeter group. The discriminant is the image of the mirrors (i.e., the invariant hyperplanes of the reflections from the group W) under the projection $V^{\mathbf{C}} \to V^{\mathbf{C}}/W$.

The coordinates in the orbit space are the elementary W-invariant polynomials in $V^{\mathbf{C}}$. For an irreducible Coxeter group, these polynomials include precisely one polynomial of maximum homogeneous degree h, which is called the Coxeter number.

Consider the projection $V^{\mathbf{C}}/W \to B = \mathbf{C}^{\mu-1}$ along the axis of the invariant of degree h, defined by elementary invariants of lower degree. Under this projection, the discriminant Φ is μ-sheeted mapped to B. Indeed, an irreducible Coxeter group contains $\mu h/2$ reflections [13]. Therefore, the W-invariant equation of the union of the mirrors is of degree μh, i.e., the equation of Φ is of degree μ relative to the invariant of degree h.

Consider Φ as the graph of a μ-valued function on B, denoting by Λ_W its 1-graph — the Lagrangian manifold in T^*B. For $W \simeq I_2(r + 2)$, Λ_W is isomorphic to the curve Λ_r: $p^2 = q^r$ on the symplectic plane T^*C.

THEOREM 14. The germ at zero of the Lagrangian map $\Lambda_W \hookrightarrow T^*B \to B$ corresponding to the irreducible Coxeter group W is simple. Conversely, let $\Lambda \hookrightarrow T^*\mathbf{C}^n \to \mathbf{C}^n$ be a simple germ of a Lagrangian map of the manifold Λ diffeomorphic to the product of germs of plane curves. Then Λ is diffeomorphic to the product of the curve Λ_r and a nonsingular manifold, and the Lagrangian map itself is equivalent to the suspension of the Lagrangian map corresponding to the irreducible Coxeter group W of type A_μ, D_μ, E_μ for r = 1, B_μ, F_4 for r = 2, H_μ for r = 3, and $I_r(r + 2)$ for r ≥ 4.

COROLLARY 1. The Legendrian manifold of the contact elements of the discriminant of an irreducible Coxeter group is isomorphic to the product of a nonsingular manifold and the curve Λ_r with r = 1 for the groups A_μ, D_μ, E_μ, with r = 2 for B_μ, F_4, and with r = 3 for H_μ.

COROLLARY 2 [27]. A germ of a vector field in V^C/W transversal to the tangent hyperplane to the discriminant at the origin is rectified by a discriminant-preserving germ of a diffeomorphism.

COROLLARY 3 [27]. The complement of the bifurcation diagram of the Lagrangian map corresponding to the irreducible Coxeter group W is the Eilenberg–MacLane space of the subgroup of index $\mu! h^\mu/\#W$ in the group of braids of μ strings.

Remark 12. a) Corollary 2 is new for the groups H_3, H_4, and Corollary 3 is new for the group H_4.

b) The proof of Theorem 14 given below is based on a classification of simple Lagrangian maps of the products of curves and its comparison with the classification of Coxeter groups. It would be interesting to find an a priori proof of this theorem or at least of Corollary 1.

Proof. A stable germ of the Lagrangian map $\Lambda \hookrightarrow T^*\mathbf{C} \to \mathbf{C}$ is of degree $\mu \le 2$ (see example in 3.4) and for $\mu = 2$ it is equivalent to the germ $\Lambda_r \hookrightarrow T^*\mathbf{C} \to \mathbf{C}$ corresponding to the Coxeter group $I_2(r + 2)$.

The degree of the projection $\Lambda^n \to C^n$ of the product of n germs of plane curves is not less than the product of the degrees of the typical projections of the factors. Therefore (see Proposition 5), if the germ $\Lambda^n \hookrightarrow T^*\mathbf{C}^n \to \mathbf{C}^n$ is simple, then Λ^n is isomorphic to the product of Λ_r and a nonsingular manifold.

The projection $(p, q) \mapsto p$ of the curve Λ_r is of degree r. This means (see Proposition 5) that for r ≥ 4 the germ of the Lagrangian map $\Lambda_r \times \mathbf{C}^{n-1} \hookrightarrow T^*\mathbf{C}^n \to \mathbf{C}^n$ at the nontransversality point $x \in \{0\} \times C^{n-1}$ of the tangent Lagrangian space to $\Lambda_r \times C^{n-1}$ and the fiber of the fibration $T^*C^n \to C^n$ is not simple.

The Lagrangian map $\Lambda_r \times \mathbf{C}^{n-1} \hookrightarrow T^*\mathbf{C}^n \to \mathbf{C}^n$ in the neighborhood of the transversality point of the tangent Lagrangian space to the fiber of the fibration is equivalent to the suspension of the Lagrangian map $I_2(r + 2)$ by Proposition 4.

This proves the theorem for Lagrangian manifolds $\Lambda_r \times C^{n-1}$ with r ≥ 4.

Let r = 1. Then the manifold $\Lambda_1 \times C^{n-1} \simeq C^n$ is nonsingular. The singularity theory of Lagrangian maps of nonsingular manifolds was constructed in [2]. By this theory, equivalence classes of simple germs of Lagrangian maps of nonsingular manifolds are in a correspondence to simple R^+-equivalence classes of germs of functions at a critical point [9]. The list of these classes, up to a stable R^+-equivalence, is the following:

$$A_\mu,\ \mu \geqslant 1:\ x^{\mu+1};\quad D_\mu,\ \mu \geqslant 4:\ x^2y + y^{\mu-1};$$
$$E_6:\ x^3 + y^4;\quad E_7:\ x^3 + xy^3;\quad E_8:\ x^3 + y^5.$$

R^+-miniversal deformations of these functions are the generating families (see Sec. 4) of the Lagrangian maps corresponding to Coxeter groups A_μ, D_μ, E_μ, respectively (see [2, 36, 43]). Simple germs of Lagrangian maps of nonsingular manifolds are equivalent to their suspensions, and conversely.

Let r = 2. Then the manifold $\Lambda_2 \times C^{n-1}$ is isomorphic to the union of two nonsingular n-dimensional manifolds intersecting without tangency along a nonsingular hypersurface in each of them. The singularity theory of Lagrangian maps of such manifolds was constructed in [40, 29]. Simple singularities in this theory correspond to simple germs of functions on a manifold with an edge or, equivalently, to simple germs of functions even in one of the variables. The latter are stably R^+-equivalent to the germs at zero of the following functions even in x (see [3]):

$$B_\mu,\ \mu \geqslant 2:\ x^{2\mu};\quad C_\mu,\ \mu \geqslant 3:\ x^2y + y^\mu;\quad F_4:\ x^4 + y^3.$$

R^+-miniversal deformations of these germs in the class of even functions are the generating families of the Lagrangian maps corresponding to the Coxeter groups B_μ, C_μ, F_4, respectively. The Lagrangian maps B_μ and C_μ are equivalent (their equivalence interchanges the components of the Lagrangian manifold $\Lambda_2 \times C^{\mu-2}$). Simple germs of Lagrangian maps of the manifolds $\Lambda_2 \times C^{\mu-2}$ are equivalent to the suspensions of B_μ, F_4, and conversely.

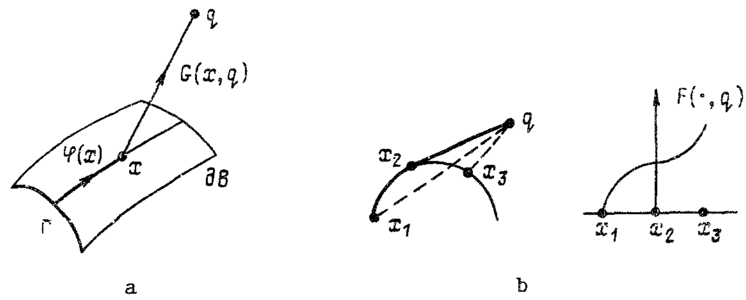


Fig. 5

Let r = 3. The curve Λ_3 is isomorphic to the open swallowtail $\Sigma_1(3)$. Simple singularities of the Lagrangian maps of the manifolds $\Lambda_3 \times C^{n-1}$ are equivalent to suspensions of the Lagrangian maps associated with the Coxeter groups H_μ and conversely. This will be proved in Sec. 10, where we describe all the simple singularities of Lagrangian maps of open swallowtails.

10. THE PROBLEM OF GOING PAST AN OBSTACLE

In Sec. 6, we associated to the bundle of geodesics γ on the boundary of an obstacle $\partial B \subset B$ the bundle of extremals in the problem of going past the obstacle which is formed by the tangents to γ oriented by the geodesics on B. The set of velocity unit vectors of the extremals in the bundle is the Lagrangian manifold $\Lambda \hookrightarrow T^*B$ on the hypersurface $\langle p, p\rangle = 1$. The set of characteristics of this hypersurface, the union of which is Λ, constitutes the ray system of the bundle γ in the space of geodesics on B. Therefore, Λ is isomorphic to the product of the ray system and the straight line.

The generating function of the Lagrangian map $\Lambda \hookrightarrow T^*B \to B$ is a multivalued function on B whose values at the point q are the distances from q to the initial manifold of the bundle γ along the extremals through q. We call this generating function the time function of the bundle of extremals in the problem of going past an obstacle.

O. P. Shcherbak used generating families (see Sec. 4) to describe the singularities of the time function in the problem of going past an obstacle outside the caustic of the bundle γ on ∂B, i.e., under the assumption that the manifold $l \subset T^*(\partial B)$ of the velocity unit vectors of the bundle γ is projected in a single-valued manner on ∂B.

Let $\Gamma \subset \partial B$ be the initial manifold of the geodesic bundle γ, $\varphi(x)$ the length of the geodesic bundle from Γ to $x \in \partial B$, G(x, q) the distance from x to the point $q \in B$ along the joining geodesic in B (Fig. 5a). Under our assumption, φ is a single-valued function, and $F(x, q) = G(x, q) + \varphi(x)$ is the generating family of the Lagrangian map $\Lambda \hookrightarrow T^*B \to B$.

Shcherbak's results are based on the observation that all the critical points of the function $F(\cdot, q)$ are of even multiplicity. The proof is shown in Fig. 5b: to the extremal in general position corresponds an A_2 critical point of multiplicity 2; the multiplicity of a more complicated critical point (corresponding to a singular extremal) is equal to double the number of A_2 points into which it decomposes under perturbation of the parameter q.

Stable R^+-equivalence of generating families induces an equivalence of the corresponding Lagrangian maps. Therefore, reduction to normal form of the Lagrangian map generated by the family F involves finding maximal subfamilies (with a nonsingular base) among R^+-miniversal deformations of germs of even multiplicity of functions in which the critical points of all the functions are only of even multiplicity. Below we give a list of these deformations of simple germs of functions, as constructed by Shcherbak.

Proposition 12 [31, 32, 10]. The maximal subfamilies with a nonsingular base of R^+-miniversal deformations of simple germs of functions are exhausted, up to stable R^+-equivalence, by the following:

$$\Xi_\mu(\subset A_{2\mu}) \quad \int_0^x (u^\mu + q_1 u^{\mu-2} + \ldots + q_{\mu-1})^2\, du,$$

$$\Omega_\mu(\subset D_{2\mu}) \quad \int_0^y (u^{\mu-1} + q_1 u^{\mu-3} + \ldots + q_{\mu-3}u + x)^2\, du + q_{\mu-2}x^2 + q_{\mu-1}x,$$

$$A_3(\subset E_6) \quad x^3 + y^4 + q_1 y^2 + q_2 y,$$

$$A_4(\subset E_8) \quad x^3 + y^5 + q_1 y^3 + q_2 y^2 + q_3 y,$$

$$H_4(\subset E_8) \quad x^3 + \int_0^y (u^2 + q_1 x + q_2)^2\, du + q_3 x.$$

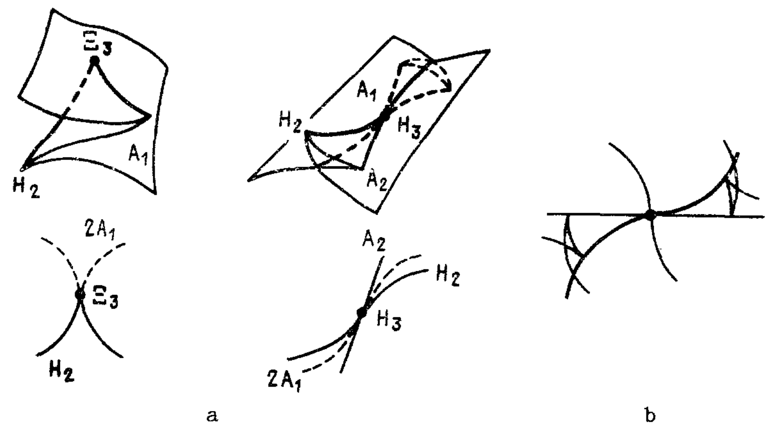


Fig. 6

Remark 13. a) The miniversal deformation of a germ $D_{2\mu}$ contains another subfamily equivalent to Ω_μ. In cases E_6 and E_8 there are also subfamilies with a singular base, while in all other cases no such subfamilies exist.

b) In the space of polynomials

$$x^{2n}+a_1x^{2n-1}+\ldots+a_{2n}$$

consider the submanifold Σ^n of polynomials with a root of multiplicity $\geq n+1$. It is isomorphic to the product of the open swallowtail $\Sigma_{n-1}(2n-1)$ and the straight line and is Lagrangian relative to the symplectic structure introduced in 5.1. The fibrations $(a_1,\ldots,a_{2n})\mapsto(a_1,\ldots,a_n)$ and $(a_1,\ldots,a_{2n})\mapsto(a_1,\ldots,a_{n-1},a_{n+1})$ are Lagrangian and define Lagrangian maps of the manifold Σ^n. It is easy to verify that these Lagrangian manifolds are equivalent to (the suspension of) Ξ_n and Ω_{n+1}, respectively.

c) The families Ξ_1, Ω_2, A_3, A_4 from Proposition 12 generate the Lagrangian maps A_1, A_2, A_3, A_4 of nonsingular manifolds. It can be shown (see [31, 32]) that the families Ξ_2, Ω_3, H_4 generate Lagrangian maps of the products of the semicubic parabola and a nonsingular manifold that correspond to the Coxeter groups H_2, H_3, H_4.

THEOREM 15. The germs at zero of the Lagrangian maps

$$\Xi_\mu\ (\mu\geqslant 3),\ \Omega_\mu\ (\mu\geqslant 4),\ H_\mu\ (\mu=2,\ 3,\ 4)$$

are simple and pairwise inequivalent. A simple germ at a singular point of the Lagrangian map of the product of the open swallowtail and a nonsingular manifold is equivalent to the suspension of one of the germs Ξ_μ, Ω_μ, H_μ.

COROLLARY 1. Simple singularities of the time function in the problem of going past an obstacle are exhausted by the singularities of the generating functions of the Lagrangian maps A_μ, D_μ, E_μ, Ξ_μ, Ω_μ, H_μ .

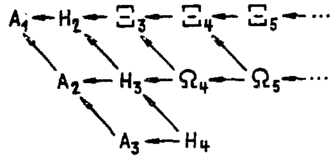
COROLLARY 2. A germ of a vector field transversal to the graph of the Lagrangian map Ξ_μ, Ω_μ, H_μ at zero is rectifiable by the germ of a graph-preserving diffeomorphism.

COROLLARY 3. The complements of complex bifurcation diagrams of the Lagrangian maps Ξ_μ, Ω_μ, H_μ are Eilenberg–MacLane spaces of a subgroup of finite index in the group of braids of μ strings.

Examples. a) Graphs and bifurcation diagrams of Ξ_3 and H_3 are shown in Fig. 6a. The self-intersection of the graphs of Ξ_μ is nonreal.

b) Figure 6b shows a family of fronts in the problem of going past an obstacle on the plane in the neighborhood of the inflection point. This family is equivalent to the family of sections of the graph H_3 by planes parallel to one of the coordinate planes.

Remark 14. a) The adjacency diagram of simple germs of Lagrangian maps of open swallowtails has the form



This follows from the similar adjacency diagram of simple even-multiplicity singularities of functions.

b) The proof of Theorem 15 given below utilizes generating families of Lagrangian maps of open swallowtails. Their construction (see Proposition 13) is based on Proposition 10, where Ξ_μ germs are reduced to normal form. Our proof essentially relies on the computations of O. P. Shcherbak (Proposition 12). An alternative proof could be devised, independent of these computations and relying on the stability theorem (Theorem 3) and it corollary on the sufficient jet of a Lagrangian map.

Proposition 13. The germ at zero of the Lagrangian map of the manifold $\Sigma_k(2k + 1) \times \mathbf{R}^{n-k}$ can be defined by the generating family

$$F(x, y_1, \ldots, y_{n-s}, q_1, \ldots, q_n) = \int_0^x (u^{k+1} + Q_1 u^{k-1} + \ldots + Q_k)^2 \, du + Q_{k+1} + y_1 q_{s+1} + \ldots + y_{n-s} q_n,$$

where s is equal to the rank of the projection of the tangent Lagrangian space to $\Sigma_k(2k + 1) \times \mathbf{R}^{n-k}$ at zero on the base of the fibration, q_i are the coordinates on the base, and Q_j are functions of $(y, q_1, \ldots, q_s)$ satisfying the conditions

$$d_0 Q_{k+1} = 0, \quad d_0^2 Q_{k+1} = 0, \quad Q_j(0) = 0, \quad \mathrm{rk}\,(\partial Q / \partial (y, q)) = k.$$

Proof. In appropriate Darboux coordinates on $T^*\mathbf{R}^n$, the tangent Lagrangian space to $\Sigma_k(2k + 1) \times \mathbf{R}^{n-k}$ at zero is given by the equations $p_1 = \ldots = p_s = q_{s+1} = \ldots = q_n = 0$. Consider the second Lagrangian fibration

$$(p, q) \mapsto (q_1, \ldots, q_s, p_{s+1}, \ldots, p_n).$$

By Proposition 10, the corresponding Lagrangian map is equivalent to the suspension of the standard map. The standard map is defined by the generating family Ξ_{k+1}. Equivalence of Lagrangian maps acts on the generating family by replacement of parameters (coordinates on the base of the Lagrangian fibration) and addition of a function of these coordinates. Therefore the second Lagrangian map is defined by the generating family

$$F(x, q_1, \ldots, q_s, p_{s+1}, \ldots, p_n) = \int_0^x (u^{k+1} + Q_1 u^{k-1} + \ldots + Q_k)^2 \, du + Q_{k+1},$$

where $(q_1, \ldots, q_s, \quad p_{s+1}, \ldots, p_n) \mapsto (Q_1, \ldots, Q_k)$ is a submersion and Q_{k+1} is a function on the base of the second fibration. Since the tangential Lagrangian space to $\Sigma_k(2k + 1) \times \mathbf{R}^{n-1}$ at zero is the zero section of the second fibration, then $d_0 Q_{k+1} = 0$ and $d_0{}^2 Q_{k+1} = 0$. Passing to the original Lagrangian fibration and setting $y_i = p_{s+i}$, we obtain the sought generating family.

Proof of Theorem 15. Let $\Sigma_k(2k+1)\mathbf{R}^{n-k} \hookrightarrow T^*\mathbf{R}^n \to \mathbf{R}^n$ be a stable Lagrangian map, L the tangent Lagrangian space to $\Sigma_k(2k + 1) \times \mathbf{R}^{n-k}$ at the point $\lambda \in \{0\} \times \mathbf{R}^{n-k}$, and F the tangent space to the fiber of the fibration at this point. If F is transversal to L, then the germ of the Lagrangian map at the point λ is equivalent to the suspension of Ξ_{k+1} (Proposition 10). Points where F is not transversal to L form in $\{0\} \times \mathbf{R}^{n-k}$ a subset X of codimension ≥ 1. Let L_1 be the hyperplane in L tangent to the cusp edge of the manifold $\Sigma_k(2k + 1) \times \mathbf{R}^{n-k}$ at the point λ. Points where F is not transversal to L_1 form a subset $Y \subset X$ of codimension ≥ 2 in $\{0\} \times \mathbf{R}^{n-k}$. We will show that the germ of a stable Lagrangian map at the point $\lambda \in X \setminus Y$ is equivalent to the suspension of Ω_{k+2}.

By Proposition 13, the germ of a Lagrangian map at the point λ can be defined by the generating family

$$F(x, y, q) = \int_0^y (u^{k+1} + Q_1 u^{k-1} + \ldots + Q_k)^2 du + Q_{k+1} + Q_{k+2} x, \tag{1}$$

where $Q_i = Q_i(x, q)$, and $Q(0) = 0, \quad d_0 Q_{k+1} = 0, \quad d_0^2 Q_{k+1} = 0, Q_{k+2}(x, 0) = 0$. Transversality of F to L_1 implies that $\partial Q_k / \partial x|_0 = A \neq 0$. Hence it follows that F(x, y, 0) is a semiquasihomogeneous function with weights deg x = (k + 1)deg y and quasihomogeneous part $\int_0^y (u^{k+1} + Ax)^2 du$. Therefore F(x, y, 0) has a $D_{2k+4}{}^+$ simple critical point at zero. The family F(x, y, q) is a deformation of F(x, y, 0) in the class of functions with critical points of even multiplicity. Therefore (Proposition 12) the family F(x, y, q) is R^+-equivalent to that induced from Ω_{k+2}:

$$\int_0^y (u^{k+1} + \hat{q}_1 u^{k-1} + \ldots + \hat{q}_{k-1} u + x)^2 du + \hat{q}_k x^2 + \hat{q}_{k+1} x.$$

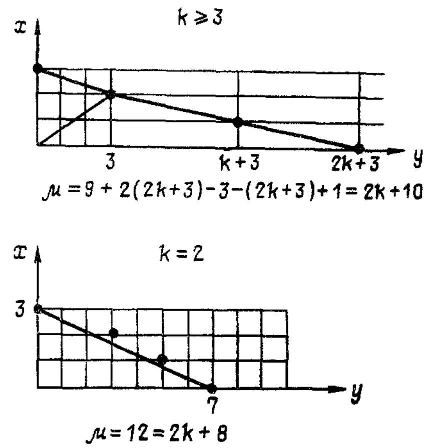


Fig. 7

Since F generates a stable Lagrangian map, the inducing map $q \mapsto q$ may be regarded as a submersion. Thus, the germ of a Lagrange map at the point λ is equivalent to the suspension of Ω_{k+2}.

Let us now show that the germ of a stable Lagrangian manifold at the point $\lambda \in Y$ is not simple for $k \geq 2$. A dense set of codimension 2 in $\{0\} \times \mathbf{R}^{n-k}$ is formed in Y by the points where $\dim F \cap L = 1$. For almost all such points, the germ of a Lagrangian map is defined by a generating family of the form (1), where

$$\partial Q_k/\partial x|_0 = 0,\ \partial Q_{k-1}/\partial x|_0 \neq 0,\ B = \partial^3 Q_{k+1}/\partial x^3|_0 \neq 0.$$

We will show that at these points the degree of the Lagrangian map is $\geq k + 4$. Since germs of this type occur irremovably in the restrictions of the Lagrangian map to $(k + 2)$-dimensional submanifolds, the Lagrangian map at the point λ is not simple (Proposition 5).

Figure 7 shows Newton's diagrams of the functions F(x, y, 0) under the above assumptions and computes their Milnor number μ using the formula of Kushnirenko [9]. Newton's diagram is determined by the highest order part $\int_0^y (u^{k+1} + Aux)^2 du + Bx^3/6$ of the function F(x, y, 0).

The degree of a Lagrangian map is equal to the number of different critical points of a general member of the generating family. Since the multiplicity of each such point is 2, the degree is $\mu/2$, i.e., at least $k + 4$.

Now let $k = 1$. At the points $\lambda \in Y$ where $\dim F \cap L = 1$, the germ of a Lagrangian map is defined by the generating family (here Q_1, Q_3 lie in the ideal (q))

$$F(x, y, q) = \int_0^y (u^2 + Q_1)^2 du + Q_2 + Q_3 x, \tag{2}$$

$Q_2 \equiv Bx^3 (\mathrm{mod}(x^4, q))$. We will show that the stable Lagrangian map defined by this generating family is equivalent to the suspension of H_4 if $B \neq 0$.

The germ $F(x, y, 0) \equiv y^5/5 + Bx^3(\mathrm{mod}(x^4))$ has an E_8 critical point. The gradient ideal of the germ F(x, y, 0) is generated by the monomials y^4, x^2. By stability of the Lagrangian map, we may take

$$Q_1 \equiv \hat{q}_1 x + \hat{q}_2 (\mathrm{mod}\,(x^2 q)), \quad Q_3 = \hat{q}_3,$$

where $q \mapsto \hat{q}$ is a submersion, i.e., $\hat{q}_1$, $\hat{q}_2$, $\hat{q}_3$ may be regarded as independent parameters of the family. Moreover, linear (in q) terms of Q_2 are divisible by x^2. Hence it follows that

$$F_{\hat{q}_1} \equiv \frac{2xy^3}{3}, \quad F_{\hat{q}_2} \equiv \frac{2y^3}{3}, \quad F_{\hat{q}_3} \equiv x\ (\mathrm{mod}\,(y^4, x^2, q)). \tag{3}$$

The monomials (3) are independent in the local algebra of singularity of F(x, y, 0).

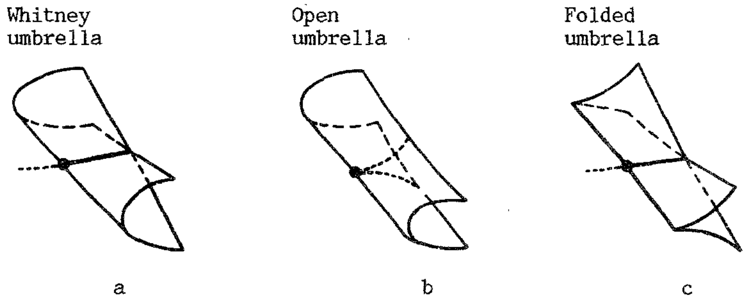


Fig. 8

Let us induce the family F from the R^+-versal deformation of the germ E_8. According to the above, the inducing map of parameter spaces is an immersion to the subfamily of functions whose critical points are all of even multiplicity. The versal deformation of the germ E_2 contains two analytic 3-parameter subfamilies with this property — A_4 and H_4. Their tangent spaces differ at zero. The formulas (3) therefore make it possible to identify our generating family with a suspension of the standard family H_4.

It remains to show that a germ of a Lagrangian map of the manifold $\Sigma_1(3) \times \mathbf{R}^{n-1}$ is not simple at the point λ, where $\dim F \cap L \geq 2$, and also if $B = 2$ in the generating family (2).

In the second case, the generating family (2) is a deformation of the germ $F(x, y, 0) \in y^5/5 + Cx^4 + (x^5)$ of multiplicity $\mu \geq 12$. In the first case, the generating family has the form

$$F(x, y, z, q) = \int_0^y (u^2 + Ax + \ldots)^2 \, du + Bz^3 + \ldots \tag{4}$$

Let $A \neq 0$, $B \neq 0$. Let $\deg y = 1/5$, $\deg x = 2/5$, $\deg z = 1/3$. Clearly, for $q = 0$, the omitted terms in (4) have $\deg > 1$. The principal part of the function $F(x, y, z, 0)$ is quasihomogeneous of $\deg = 1$ and is the direct sum of a D_6^+ singularity in the variables x, y and a A_2 singularity in the variable z. Therefore, the critical point of the germ $F(x, y, z, 0)$ is also of multiplicity 12.

Denote by $Z \subset Y$ the set of points λ at which the germ of a stable Lagrangian map is not of type H_4. The codimension of Z in $\{0\} \times \mathbf{R}^{n-1}$ is 3. Therefore the points of Z occur irremovably in the restrictions of the Lagrangian map to 4-dimensional submanifolds. On the other hand, a dense set in Z is formed by the points of the two types considered above, at which the degree of the Lagrangian map is 6. By Proposition 5, the germs of a Lagrangian map at points from Z are not simple.

11. RAY SYSTEMS IN GENERAL POSITION, OPEN WHITNEY UMBRELLAS, AND THE TOPOLOGY OF LAGRANGIAN SURFACES

11.1. In the space with the coordinates $(x, \lambda_1, \ldots, \lambda_n, \delta_0, \ldots, \delta_n)$ consider a submanifold of codimension 2 defined by the equations

$$F = x^{n+1} + \lambda_1 x^{n-1} + \ldots + \lambda_n = 0,$$
$$G = \delta_0 x^n + \delta_1 x^{n-1} + \ldots + \delta_n = 0.$$

The projection $(x, \lambda, \delta) \mapsto (\lambda, \delta)$ of this submanifold is the map $\mathbf{R}^{2n} \to \mathbf{R}^{2n+1}$, called the Morin map. The image of the Morin map is called a generalized Whitney umbrella. For $n = 1$, this image is the surface in $\mathbf{R}^3$ with the equation $\delta_1^2 = \delta_0^2 \lambda_1$, i.e., an ordinary Whitney umbrella (Fig. 8a). The kernel of the Morin map at singular points is one-dimensional and is generated by the vector $\partial/\partial x$. It can be shown [38] that the singularities of corank 1 of the map $\mathbf{R}^N \to \mathbf{R}^{N+1}$ in general position are exhausted by the singularities of Morin maps up to left-right equivalence and suspension.

Denote by $\mathscr{A}$ the subalgebra of the functions from $\mathcal{O}_{\mathbf{R}^{2n}}$, whose derivative in the direction of the kernel of the Morin map $\mathbf{R}^{2n} \to \mathbf{R}^{2n+1}$ vanishes at the singular points. The algebra $\mathscr{A}$ is the algebra $\mathcal{O}_{2n}$ of regular functions of some analytic set Λ^{2n}.

Definition. The manifolds Λ^{2n} are called open Whitney umbrellas.

THEOREM 16. The open Whitney umbrella Λ^{2n} is isomorphic to the following manifolds:

1) the conormal fiber space of the open swallowtail $\Sigma^n \subset \mathbf{R}^{2n}$;

2) the submanifold in the space of pairs of polynomials

$$F = \frac{x^{2n+1}}{(2n+1)!} + q_1 \frac{x^{2n-1}}{(2n-1)!} + \ldots + q_{2n} \frac{x^0}{0!},$$
$$G = (-1)^{2n} p_{2n} \frac{x^{2n-1}}{(2n-1)!} + \ldots + (-1)^1 p_1 \frac{x^0}{0!},$$

formed by pairs with a common root of multiplicity not lower than (n + 1, n), respectively;

3) one of the two components of the Lagrangian manifold generated by the family of functions of x,

$$\mathcal{F}_n(x, q, Q) = \int_0^x (Q_1 \xi^{n-1} + \ldots + Q_n)(\xi^{n+1} + q_1 \xi^{n-1} + \ldots + q_n)\, d\xi$$

(the equation $\mathcal{F}_x = 0$ of the critical points of the family decomposes into two equations; the relevant component corresponds to the equation $x^{n+1} + q_1 x^{n-1} + \ldots + q_n = 0$).

COROLLARY. An open Whitney umbrella is a Lagrangian manifold.

Remark 15. a) The strata of singularities of the manifold Λ^{2n} form a flag of manifolds of even dimension (Fig. 8b). The 2k-dimensional stratum is isomorphic to Λ^{2k}.

b) (V. M. Zakalyukin) An open umbrella in the space of pairs of polynomials is Lagrangian relative to the symplectic structure $\Sigma dp_i \wedge dq_i$. This can be proved using a canonical symplectic structure in the cotangent fiber space of the space of binary forms of degree 2n + 1.

c) The graph of the Lagrangian map generated by the family $\mathcal{F}_1$ is the surface in $\mathbf{R}^3$ with the equation $9u^2 + 4Q_1^2 q_1^3 = 0$ (Fig. 8c). It was apparently first considered in [5, 30].

Proof of Theorem 16. 1°. The open swallowtail $\Sigma_n(2n + 1)$ is Lagrangian in the polynomial space $\mathcal{P}_{2n+1}$. Therefore its conormal fibration is isomorphic to the fibration of tangent Lagrangian spaces. Using the parametrization

$$\pi\colon\ (a, b, \ldots) \mapsto (x - a)^{n+1}(x^n + (n+1)ax^{n-1} + bx^{n-2} + \ldots)$$

of the manifold $\Sigma_n(2n + 1)$, we can easily show that the tangent space at the nonsingular point $\pi(a, b, \ldots)$ consists of polynomials of degree 2n − 1 for which a is a root of multiplicity n.

2°. Let $P_i = \partial\mathcal{F}_n/\partial Q_i$, $p_i = \partial\mathcal{F}_n/\partial q_i$ and consider two maps $(x, q_1, \ldots, q_{n-1}) \mapsto (P, q)$ and $(x, Q) \mapsto (Q, P)$. The image of the first map is the open swallowtail $\Sigma_n(2n + 1)$ in the space of the polynomials F_{2n+1}, and x is a root of multiplicity (n + 1) of the corresponding polynomial (see the lemma in the proof of Theorem 10). For a fixed x, the image of the second map consists of the polynomials G_{2n-1} for which x is a root of multiplicity n.

3°. The generating family $\mathcal{F}_n$ has the form $\int_0^x G'(\xi, Q) F(\xi, q)\, d\xi$ (up to notation). On the normalization $\tilde{\Sigma}$ of the corresponding Lagrangian manifold Σ, the functions $(Q_1, \ldots, Q_n, q_1, \ldots, q_{n-1}, x)$ may be used as the coordinates.

LEMMA. 1) $\mathcal{O}_\Sigma$ is the subspace $\mathcal{L} \subset \mathcal{O}_{\tilde{\Sigma}}$ of functions of the form $\int_0^x (AF' + BG')\, d\xi + C$, where $A, B, C \in \mathcal{O}_{\tilde{\Sigma}}$, $\partial C/\partial x = 0$.

2) $\mathcal{O}_\Sigma$ is generated as a $\mathcal{O}_{(Q,q)}$-module by the functions $(1, p_1, \ldots, p_n, P_1, \ldots, P_n)$.

Indeed, the inclusion $\mathcal{O}_\Sigma \subset \mathcal{L}$ is obvious. We will show that $\mathcal{L}$ is contained in the $\mathcal{O}_{(Q,q)}$-module generated by $1, p_j, P_j$. Divide B by F′: $B = RF' + \sum_{i=1}^n D_i(Q, q)\, x^{n-i}$. The function $\int_0^x (\Sigma D_i \xi^{n-i})\, G'\, d\xi$ lies in the $\mathcal{O}_{(Q,q)}$-submodule generated by $P_1, \ldots, P_n$. The function $\int_0^x (A + RG') F'\, d\xi + C$ lies in the $\mathcal{O}_{(Q,q)}$-submodule generated by $1, p_1, \ldots, p_n$ (by Theorem 10).

4°. The map of the normalization $\tilde{\Sigma}$ defined by the functions $(Q_1, \ldots, Q_n, q_1, \ldots, q_n, p_n)$ is a Morin map. Its critical points satisfy the equations $F'(x, q) = G'(x, q) = 0$. Therefore the space $\mathscr{L} \subset \mathcal{O}_{\tilde{\Sigma}}$ is precisely the algebra of regular functions on the open umbrella Λ^{2n}. By the above lemma, $\Sigma = \Lambda^{2n}$.

11.2. Let a nonsingular hypersurface H^{2n+1} be defined in the symplectic space (M^{2n+2}, ω). Denote by N^{2n} the symplectic manifold of the characteristics of the hypersurface H.

Definition. The initial manifold of a ray system is the n-dimensional nonsingular isotropic manifold l^n on the hypersurface H. The manifold $\Lambda \subset N$ of the characteristics through ℓ is called a ray system.

In general, a ray system is a Lagrangian manifold in N.

An initial manifold that belongs to a prespecified open dense subset in the space of embeddings $l \hookrightarrow H$ (not necessarily isotropic) will be called typical and its ray system will be called a ray system in general position.

THEOREM 17. A germ of a ray system in general position is symplectically isomorphic to a germ of an open Whitney umbrella.

COROLLARY. Initial manifolds of ray systems whose singularities are only open umbrellas and transversal intersections form an open set in the C^∞-topology of the space of all initial manifolds.

Conjecture. This open set is dense in the space of initial manifolds.

Remark 16. Ray systems usually arise as solutions of Hamilton–Jacobi equations. Such an equation is defined by the hypersurface $H \subset T^*B$. The solution with the initial condition φ: $\partial B \to \mathbf{R}$, ∂B a hypersurface in B, is constructed as follows. The initial manifold l is defined as the intersection with H of the Lagrangian manifold L of the covectors on B applied at the points of ∂B which coincide with $d\varphi$ when restricted to ∂B. The union of the H-characteristics through l is a Lagrangian manifold in T^*B. The corresponding multivalued function is the solution of the Hamilton–Jacobi equation with the single-valued initial condition φ. However, for ∂B and φ in general position, L intersects transversally with H. It can be shown that the ray system with the initial manifold $l = L \cap H$ is an immersed Lagrangian manifold. Thus, open umbrellas do not arise as typical singularities of ray systems in the Cauchy problem for the Hamilton–Jacobi equation if the initial condition φ is single-valued. This construction of a ray system is in fact a particular case of the specification of a Lagrangian manifold using a generating family. We will show (see 11.4) that the open Whitney umbrella Λ^{2n} in itself, without extra components, cannot be defined at all by a generating family.

Proof of Theorem 17. In a local situation, we may regard H as a hyperplane in a linear space. Define the Gaussian map $l \to G$ to the Grassmann manifold G of n-dimensional isotropic subspaces in H, associating to the point $\lambda \in l$ the tangent space $T_\lambda l$. The following lemma is easily verified.

LEMMA 1. The manifold G is nonsingular. Isotropic n-dimensional subspaces in H containing a characteristic direction form a nonsingular submanifold $\Sigma \subset G$ of codimension 2.

For a typical initial manifold l, the Gaussian map is transversal to Σ. The preimage of Σ in l is the submanifold l' of codimension 2. The field of directions tangential to l is defined at the points of l' — this is the field v of the kernels of the projection $l \hookrightarrow H \to N$. Extend it to the field of directions on the entire l. For a typical initial manifold, the projection of l' along the integral curves of this field is a Morin map (it is easy to see that this property depends on v, and not on its extension). Therefore, the field v is reducible to the normal form $\partial/\partial x$ in the coordinates $(\lambda_1, \ldots, \lambda_{k-1}, \delta_0, \ldots, \delta_{k-1}, x, y_1, \ldots, y_{n-2k})$ on l in which l' is defined by the equations $x^k + \lambda_1 x^{k-2} + \ldots + \lambda_{k-1} = \delta_0 x^{k-1} + \ldots + \delta_{k-1} = 0$. Therefore the subalgebra $\mathscr{A} \subset \mathcal{O}_l$ of the functions whose derivative along v vanishes on l' is isomorphic to the algebra $\mathcal{O}_{\Lambda^{2k} \times D^{n-2k}}$.

Let Δ be a ray system of the initial manifold l. Then $\mathcal{O}_\Delta \subset \mathscr{A}$. For a typical l, this inclusion is an equality. Therefore Δ is diffeomorphic to the germ at zero of the manifold $\Lambda^{2k} \times D^{n-2k}$.

Lemma 2 concludes the proof of the theorem.

LEMMA 2. The symplectic structure ω in which Λ^{2k} is Lagrangian is reducible to the normal form of Theorem 16.3.

We will show that the 0-jet of the form ω is homotopic to the 0-jet of the normal form $\pm\omega_0$ in the class of 0-jets of symplectic structures that are zero on Λ^{2k}. This would imply Lemma 2 by the results of Sec. 1 and the fact that the structures $\pm\omega_0$ are transformed one into another by quasihomogeneous stretchings.

Define a quasihomogeneous structure in the space $\mathbf{R}^{4k} \supset \Lambda^{2k}$, setting $\deg q_i = \deg Q_i = i+1$, $\deg p_i = \deg P_i = 2k+2-i$. The manifold Λ^{2k} is quasihomogeneous and Lagrangian in the symplectic structure $\omega_0 = \Sigma (dp_i \wedge dq_i + dP_i \wedge dQ_i)$ of degree $2k + 3$. On the other hand, Λ^{2k} is the conormal fiber space of the open swallowtail $\Sigma^k \subset \mathbf{R}^{2k}$. Let $\omega_1 = \Sigma dP_i \wedge dq_i$ be a quasihomogeneous symplectic 2-form in $\mathbf{R}^{2k}$ in which Σ^k is Lagrangian. Then $\omega_0 + \lambda\omega_1$ is a quasihomogeneous symplectic structure in $\mathbf{R}^{4k} = T^*\mathbf{R}^{2k}$ which is zero on Λ^{2k}.

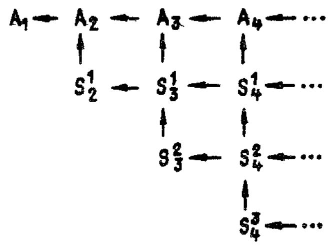


Fig. 9

Let $\mathfrak{L}$ be the space of 0-jets of closed 2-forms in $\mathbf{R}^{4k}$ that are zero on Λ^{2k}, $\mathscr{L} = \underset{d}{\oplus} \mathscr{L}_d$. its quasihomogeneous gradation. To prove the lemma, it suffices to show that $\mathscr{L}_d = 0$. for $d < 2k+3$, $\mathscr{L}_{2k+3} = \langle \omega_0, \omega_1 \rangle$. On the normalization of the manifold Λ^{2k} we have in the coordinates $(x, q_1, \ldots, q_{k-1}, Q_1, \ldots, Q_k)$ (taking $P_{k+1} \triangleq q_k$)

$$\begin{aligned} dP_i \wedge dq_j &\sim q_{k-1} x^{k+1-i} dx \wedge dq_j + \ldots \\ dp_i \wedge dq_j &\sim Q_k x^{k-i} dx \wedge dq_j + \ldots \end{aligned} \quad (j < k)$$
$$\begin{aligned} dP_i \wedge dQ_j &\sim q_{k-1} x^{k+1-i} dx \wedge dQ_j + \ldots \\ dp_i \wedge dQ_j &\sim Q_k x^{k-i} dx \wedge dQ_j + \ldots \end{aligned} \quad (j \leqslant k) \tag{1}$$

The monomials in the left-hand side are of degree $2k + 3 + j - i$. It is easy to verify that for $i > j$ the monomials in the right-hand side do not cancel with anything in $\Omega^2(\Lambda^{2k})$, because $\deg P_\alpha \geq k + 1$, $\deg p_\beta \geq k + 2$. Therefore the space of 2-forms generated by the monomials in the left-hand side and the monomials $dq_i \wedge dq_j$ $(i, j < k)$, $dq_i \wedge dQ_j$ $(i < k)$, $dQ_i \wedge dQ_j$, is disjoint with $\mathfrak{L}$.

Thus, $\mathfrak{L}_d = 0$ for $d < 2k + 3$. For the same reasons, the space $\mathfrak{L}_{2k+3}$ does not contain linear combinations of the 2-forms $dp_i \wedge dQ_j$. For the remaining 2-forms (1) with $i = j$, the monomials in the right-hand side occur in expansions of forms from $\Omega^2(\lambda^{2k})$ precisely one more time — in expansions of the form $dP_k \wedge dq_k$ or $dp_k \wedge dq_k$. Therefore, $\dim \mathfrak{L}_{2k+3} = 2$.

11.3. Consider the generating family of the functions of x

$$\mathscr{F}^k_{n+1} = \int_0^x (\xi^{n+1} + q_1\xi^{n-1} + \ldots + q_n)(\xi^k + Q_1\xi^{k-1} + \ldots + Q_k)\, d\xi, \quad n \geqslant k.$$

It is easy to verify that one of the two components of the Lagrangian manifold generated by this family (specifically, the component corresponding to the roots of the equation $x^{n+1} + q_1x^{n-1} + \ldots + q_n = 0$) is isomorphic to the product of the open Whitney umbrella Λ^{2k} and a nonsingular $(n - k)$-dimensional manifold. Denote the germ at zero of its Lagrangian map to the (q, Q)-space by $S_{n+1}{}^k$. The Lagrangian map $S_{n+1}{}^0$ is equivalent to A_{n+1}. The Lagrangian map of the manifold Λ^{2n} described by Theorem 16.3 is equivalent to $S_{n+1}{}^n$.

THEOREM 18. 1) The Lagrangian maps $S_{n+1}{}^k$ are simple.

2) The Lagrangian map in general position of the open Whitney umbrella Λ^{2n} is equivalent in the neighborhood of the origin to the germ $S_{n+1}{}^n$.

Proof. The lemma in the proof of Theorem 16 implies versality of the Lagrangian map $S_{n+1}{}^n$. It is therefore stable. We similarly check stability of the germs $S_{n+1}{}^k$ for $k < n$. In the coordinates $(x, q_1, \ldots, q_{n-1}, Q_1, \ldots, Q_k)$ on the normalization of the Lagrangian manifold generated by the family $\mathscr{F}^k_{n+1} = \int_0^x F_{n+1} G_k d\xi$, the equations $F'(x) = F''(x) = \ldots = F^{(s)}(x) = 0$ define the stratum of A_{s+1} singularities of the Lagrangian map $S_{n+1}{}^k$, and the equations $F'(x) = \ldots = F^{(r)}(x) = G(x) = \ldots = G^{(r-1)}(x) = 0$ define the stratum of singularities of codimension 2r of the Lagrangian manifold being mapped. Hence follows the adjacency diagram of the classes of the stable equivalence $S_{s+1}{}^r$ (Fig. 9) and its simplicity.

Let us now prove part 2 of the theorem.

The germ $S_{n+1}{}^n$ has a sufficient 1-jet by Corollary 2 of Theorem 3. Therefore, it suffices to verify that almost all Lagrangian spaces applied at the origin are transformed one into another by symplectomorphisms preserving Λ^{2n}. Consider the action on a Lagrangian Grassmannian of the group of the linear parts of these symplectomorphisms. The tangent space to the Lagrangian Grassmannian at a point is identified with the space of quadratic forms on the Lagrangian space corresponding to this point. The velocity vector of the flow defined by a quadratic Hamiltonian is the Hamiltonian restriction to this Lagrangian

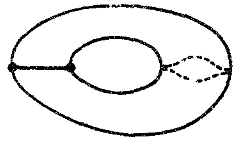


Fig. 10

space. By versality of Lagrangian map S_{n+1}^n, for every i, j = 1,..., 2n there exists a Hamiltonian $p_i p_j - \Sigma a_{ij}^k(q)p_k - b_{ij}(q)$, whose flow preserves Λ^{2n}. Comparing the quasihomogeneous degrees of the terms, we easily verify that $a_{ij}^k \in \mathfrak{m}$, $b_{ij} \in \mathfrak{m}^3$. Therefore, the quadratic parts of the Hamiltonians vanish on the subspace $p^{-1}(0)$ and form in the Lie algebra of quadratic Hamiltonians a stationary subalgebra of this subspace. The sought proposition now follows from the fact that a stationary subgroup of a Lagrangian subspace acts transitively on the set of Lagrangian spaces transversal to this subspace.

Remark 17. a) Lagrangian maps generated by the families $\mathcal{F}_n^k$ (for any k, n) were first studied by Zakalyukin [23]. He examined a Lagrangian manifold consisting of two components $\Lambda^{2d} \times \mathbf{R}^{n+k-1-2d}$ and $\Lambda^{2(d-1)} \times \mathbf{R}^{n+k+1-2d}$, $d = \min(n, k)$. Applying Theorem 3, we can easily show that the Lagrangian maps of these manifolds generated by the families $\mathcal{F}_n^k$ are simple at least on the formal series level.

b) For k > 0, the graphs of the Lagrangian maps S_{n+1}^k are called folded Whitney umbrellas (for k = 0, these are generalized swallowtails). The simplest of them (S_2^1) is shown in FIg. 8c. We still do not know if the list $\{S_{n+1}^k\}$ of simple classes of Lagrangian maps of the manifolds $\Lambda^{2k} \times D$ is complete.

c) The Lagrangian map S_{n+1}^k to a (n + k)-dimensional space is of degree n + 1 and multiplicity n + k + 1. Therefore, Theorems 4 and 5 are inapplicable to this map for k > 0. Nevertheless, the complement of the complex bifurcation diagram of the germ S_2^1 is the Eilenberg–MacLane space of the group $\mathbf{Z} \oplus \mathbf{Z}$. Theorem 6 is also applicable to the Lagrangian maps S_{n+1}^k. Therefore a vector field transversal to a folded Whitney umbrella at the origin is rectifiable by a local umbrella-preserving diffeomorphism.

11.4. The map i: $l^2 \to (M^4, \omega)$ of a surface in a symplectic manifold is called isotropic if $i^*\omega = 0$. An isotropic map for which all the singularities of the image $\Lambda = i(l)$ are open Whitney umbrellas and transversal self-intersections will be called a Lagrangian imbedding. In the space of isotropic maps of a closed surface, Lagrangian imbeddings form an open (Theorem 17) nonempty (see [20]) and apparently dense subset. Note that a Lagrangian imbedding in the neighborhood of a singular point of an open umbrella is a homeomorphism on its image. Lagrangian imbeddings without self-intersection points of all closed surfaces with Euler characteristic $\chi \leq -2$ into the standard symplectic space $\mathbf{R}^4$ were described in [20].

For the Lagrangian imbedding i: $l^2 \to (M^4, \omega)$ of a closed surface we have the equality (modulo 2 for a nonorientable l) [20]

$$l \cdot l = \chi(l) + 2\varkappa + \lambda, \tag{2}$$

where $l \cdot l$ is the self-intersection index of the fundamental cycle $i_*[l]$ in $H_2(M)$, χ is the Euler characteristic of the surface, $\varkappa$ is the number of self-intersection points (counting the signs as determined by the orientation of l), λ is the number of open umbrellas Λ^2. A similar formula for immersions ($\lambda = 0$) is well known. From this formula it follows, for example, that surfaces with odd χ cannot be Lagrangian-immersed in $\mathbf{R}^4$.

It is interesting to trace the effect of the singularities of Λ^2 on the self-intersection points of exact Lagrangian imbeddings (see Sec. 2). Gromov's theory on the existence of self-intersection points for exact immersions is apparently also true for exact imbeddings. According to the general philosophy of symplectic topology, the number of self-intersection points of a closed manifold exactly immersed in $T^*\mathbf{R}^n$ is not less than half the number of critical points that a Morse function can have as a minimum on this manifold. Figure 10 is the graph of the Lagrangian map $\Lambda \hookrightarrow T^*\mathbf{R}^2 \to \mathbf{R}^2$ of an exactly imbedded Klein bottle with $\lambda = 2$ and = 1 (from the previous considerations, we should have $\varkappa \geq 2$). Analysis of such examples creates the impression that singular points of a Lagrangian imbedding may "swallow" the saddle points, without "touching" the maxima and the minima of a function.

To the singular point Λ^2 of an oriented Lagrangian-imbedded surface we associate the sign of the Maslov index of the closed curve going around the singular point in the positive direction. This index equals ±2 depending on the orientation, which is easily calculated from Fig. 8c. Hence it follows that on a closed oriented surface Lagrangian-imbedded in $\mathbf{R}^4$, the number of positive open umbrellas is equal to the number of negative open umbrellas. It is remarkable that both make a positive contribution to the form (2).

On a Lagrangian manifold defined by a generating family, the Maslov index of a closed curve not passing through the singularities of the manifold is defined and equal to zero. Hence it follows that the open umbrella Λ^{2n} cannot be defined by a generating family.

APPENDIX

1. Versality Theorem for Semiforms on the Straight Line

THEOREM [19]. The analytic family of forms $F(x, \lambda)(dx)^{\alpha}$, where $\alpha = 1/k$, $k \in N$, and the germ at zero of the function $F(\cdot, 0)$ is R-equivalent to $\pm x^{\mu+1}$, is representable as $F_0(X, \lambda)(dx)^{\alpha}$ in appropriate local analytic coordinates $X = X(x, \lambda)$, $\Lambda = \Lambda(\lambda)$, where

$$F_0(X, \Lambda) = \pm (X^{\mu+1} + \Lambda_1 X^{\mu-1} + \ldots + \Lambda_\mu).$$

Proof. The family of functions $G(x, \lambda) = \int_0^x F^k(\xi, \lambda)\, d\xi$ may be regarded as a deformation of the germ at zero of the function $G(x, 0) = \int_0^x F^k(\xi, 0)\, d\xi$ of multiplicity $k(\mu + 1)$. By the versality theorem for functions [9], it is R^+-equivalent to the family induced from the R^+-versal deformation

$$H = \pm \int_0^x (\xi^{k(\mu+1)} + A_2 \xi^{k(\mu+1)-2} + \ldots + A_{k(\mu+1)})\, d\xi$$

of such a germ. The image of the inducing map $\lambda \mapsto A$ lies in the space of parameter values for which the critical points of the function $H(\cdot, A)$ are of multiplicity $\geq k$. In the analytic case, this space consists of polynomials of the form $G_0 = \int_0^x F_0^k(\xi, \Lambda)\, d\xi$, and the map $\Lambda \mapsto A$ is a diffeomorphism on its image. Therefore the family of functions G is R^+-equivalent to that induced from G_0, and the family of 1-forms $F^k(x, \lambda)dx$ is R-equivalent to that induced from $F_0{}^k(X, \Lambda)dX$.

Remark. We have used this theorem with $\alpha = 1/2$ several times. Kostov [24] and Lando [25] generalized it to the case of arbitrary degrees of volume forms. The theorem also holds in the C^∞-category (Kostov–Lando, and independently V. V. Lychagin).

2. All Real Analytic Results of the Paper Hold in the C_∞-Category. We used analyticity in the following cases.

a) In the definition of the singular Lagrangian (Legendrian) manifold we assume that the nonsingular points are dense in an analytic manifold.

b) In the definition of the degree of the Lagrangian map as the number of its complex sheets and in the nonsimplicity criterion associated with this concept (Proposition 5).

c) In the proof of Theorem 15 for the reduction of Lagrangian maps to the normal forms Ω_μ, H_μ.

d) In the proof of the versality theorem for semiforms.

In the smooth case, the definition of Lagrangian and Legendrian manifolds should be sharpened by considering, e.g., only sets that are diffeomorphic to analytical sets. After this modification, the proofs of all general theorem carry over verbatim to the C^∞-category. In particular, this relates to the sufficient jet theorem (corollaries of Theorem 3). The degree of a stable Lagrangian map can be defined as the degree of its sufficient jet. Therefore the criterion of nonsimplicity of a Lagrangian map remains valid in the smooth case also. Reduction of smooth Lagrangian maps to normal forms is converted to the analytic case by Corollary 3 of Theorem 3. The versality theorem for semiforms is also true in the C^∞-category, as we have noted in Appendix 1.

Remark. The Weierstrass preparation theorem, which has been repeatedly applied in the C^∞-case, is replaced with the Weierstrass–Malgrange theorem [9]. However, the function expansions used in the proof of Theorem 4 on vector fields may be multivalued in the smooth case. Therefore, the above does not apply to Theorem 4 in the holomorphic category. This theorem is also valid in the real analytic case, but in general it is not true in the C^∞-category (see [22, 27]).

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