The Mirror Formula for Quintic Threefolds

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To D. B. Fuchs on his 60-th anniversary

The formula of Candelas, de la Ossa, Green and Parkes [6] expressing the virtual numbers n_d, $d = 1, 2, 3, ...$, of degree d holomorphic spheres on quintic threefolds in $\mathbb{C}P^4$ in terms of series solutions to the linear differential equation

$$(q\frac{d}{dq})^4 \;\; I = 5q \;(5q\frac{d}{dq}+1)(5q\frac{d}{dq}+2)(5q\frac{d}{dq}+3)(5q\frac{d}{dq}+4) \;\; I \qquad (1)$$

has been intriguing algebraic and symplectic geometers since the beginning of the decade. The first proof of this formula was given two years ago in the extensive paper [9] among a number of other theorems on equivariant Gromov – Witten theory. Several authors managed to adjust the approach of that paper to complete intersections in homogeneous Kähler spaces [13, 2, 14], in toric manifolds [11, 12] and to symmetric products of Riemann surfaces [4].

We present here a shortcut to our original proof in the case of quintic threefolds. Several variants of the proof can be found in [9, 11, 12, 17, 5, 20] as particular cases of more general theorems. Yet it seems useful to illustrate all ingredients of the proof in the simplest nontrivial example.

We will assume that the reader is familiar with generalities on orbifolds and orbibundles, equivariant cohomology and localization formulas, Kontsevich's moduli spaces of stable maps [15, 3] and with the formulation of the conjecture. We will concentrate therefore only on the issues relevant for the proof of the mirror formulas.

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In the last section named *Updates* we outline several modifications of the proof available at the moment and provide corresponding references.

The linear sigma-model. A quintic threefold in $X = \mathbb{C}P^4$ is given by a generic degree 5 homogeneous equation $Q(x_1, ..., x_5) = 0$. A degree d parametrized curve $\mathbb{C}P^1 \to X$ is described by 5 relatively prime degree d binary forms which we will represent by the polynomials $x_1(\zeta), ..., x_5(\zeta)$ of degree $\leq d$ in the affine coordinate ζ on $\mathbb{C}P^1$. The curve is situated on the quintic if and only if the degree $\leq 5d$ polynomial $Q(x_1(\zeta), ..., x_5(\zeta))$ vanishes identically. This identity yields $5d + 1$ equations of degree 5 in the projective space $LX_d = \mathbb{C}P^{5d+4}$ of all (not necessarily relatively prime) 5-tuples of degree $\leq d$ polynomials. Attempting to count degree d spheres on the quintics by means of intersection theory in LX_d we will arrive to the answer $n_d = 5^{5d+1}$ which is meaningful, as it has been explained by D. Morrison and R. Plesser in [19], but wrong.

Our approach to the quintic formula begins with the following observation [10]. The variety given in LX_d by the equations $Q(x(\zeta)) \equiv 0$ is invariant with respect to the Möbius transformation group of $\mathbb{C}P^1$, and one can employ *equivariant* intersection theory for curve counting. The maximal torus S^1 in the Möbius group acts via $\zeta \mapsto \zeta \exp i\phi$ on the space LX_d. The cohomology algebra $H^*_{S^1}(LX_d)$ is generated by the equivariant Chern class $-p$ of the Hopf bundle over LX_d. The polynomial Q defines an invariant section $Q(x(\zeta))$ of an equivariant $5d + 1$-dimensional bundle LV_d over LX_d. The following formal series encodes complete equivariant topological information about equivariant Euler classes of these bundles:

$$L(q, z) := \sum_{d=0}^{\infty} q^d \int_{LX_d} e^{pz} Euler(LV_d) \ .$$

The integral in this formula means evaluation of an equivariant cohomology class on the invariant fundamental class of the manifold and takes values in the coefficient algebra $H^*(BS^1)$ of the equivariant cohomology theory. Thus the coefficients of the (q, z)-series L are polynomials in one variable which we prefer to denote $\hbar$. With this notation $Euler(LV_d) = 5p(5p - \hbar)(5p - 2\hbar)...(5p - 5d\hbar)$.

Theorem A ([10]). *(a)*

$$L(q, z) = \langle I(qe^{\hbar z}, \hbar^{-1}), I(q, -\hbar^{-1}) \rangle \ ,$$

where

$$\langle \phi, \psi \rangle := \int_{\mathbb{C}P^4} \phi \ \psi \ Euler(\mathcal{O}(5)) = \frac{1}{2\pi i} \oint \phi(P)\psi(P) \frac{5PdP}{P^5}$$

is the intersection pairing in the even cohomology algebra $\mathbb{Q}[P]/(P^4)$ *of the quintic, and*

$$I(q, \hbar^{-1}) := e^{P \ln q/\hbar} \sum_{d=0}^{\infty} q^d \frac{(5P+\hbar)(5P+2\hbar)...(5P+5d\hbar)}{(P+\hbar)^5(P+2\hbar)^5...(P+d\hbar)^5}$$

is a formal vector-function with coefficients in this algebra.

(b) Components of the vector-function form a fundamental solution to the linear differential equation (1).

Proof. (a) Compute the coefficients of the series L explicitly by the Duistermaat – Heckman formula,

$$\int_{LX_d} \Phi(p, \hbar) = \frac{1}{2\pi i} \oint \frac{\Phi(p, \hbar)dp}{p^5(p-\hbar)^5...(p-d\hbar)^5},$$

and change the order of summation. (b) Substitute I into (1). $\square$

The non-linear sigma-model. The definition of the numbers n_d accepted in [9] was given by M. Kontsevich [15] on the basis of intersection theory in moduli spaces of stable maps to X. Let $X_{n,d}$ denote the moduli orbifold of degree d genus 0 stable maps with n marked points. For any such map $f : (\Sigma, \varepsilon_1, ..., \varepsilon_n) \to X$, the section Q of the degree 5 line bundle over X defines an element in the $5d+1$-dimensional space $H^0(\Sigma, f^*\mathcal{O}(5))$ and hence — a section of the orbibundle V_d over $X_{n,d}$ formed by these spaces. According to Kontsevich's definition the Euler class of this bundle capped with the fundamental class of the orbifold is taken on the role of the virtual fundamental class in the corresponding space of stable maps to the quintic. Taking in account the multiple cover formula [1] proved in [18], one gives the following recursive definition of the virtual numbers of degree d spheres in quintic threefolds

$$\sum_{m|d} \frac{n_{d/m}}{m^3} := \int_{X_{0,d}} Euler(V_d).$$

The first step in our proof of the conjecture extracting these numbers from the differential equation (1) consists in mimicking Theorem A within the framework of moduli spaces.

Introduce the *graph space* $GX_{n,d}$ as the moduli orbifold of stable maps to $GX = \mathbb{C}P^1 \times X$ of degree d in projection to X and degree 1 in projection to $\mathbb{C}P^1$. Let GV_d be the orbibundle with the fiber $H^0(\Sigma, f^* \ \mathcal{O}_{\mathbb{C}P^1} \otimes \mathcal{O}_X(5))$ of dimension $5d+1$. The space and the bundle inherit the S^1-action from the Möbius transformation group on $\mathbb{C}P^1$.

There is a natural equivariant birational isomorphism (see Main Lemma in [9])

$$\mu : GX_{0,d} \to LX_d \tag{2}$$

defined as follows.

A stable map $f : \Sigma \to \mathbb{C}P^1 \times X$ of bi-degree $(1,d)$ is represented by the graph of a map $f_0 : \mathbb{C}P^1 \to X$ of some degree $d_0 \leq d$ and several "vertical" curves $f_i : (\Sigma_i, \varepsilon_i) \to \{\zeta_i\} \times X$ of bi-degrees $(0, d_i)$, $i = 1, ..., r$, attached to the graph at the points $f_i(\varepsilon_i)$. The number r of such vertical curves can vary from 0 to d, and their total degree $d_1 + ... + d_r$ equals $d - d_0$.

The graph component f_0 is described by 5 mutually prime binary forms of degree d_0 uniquely up to a common constant factor (we record them by 5 polynomials $(x_1'(\zeta), ..., x_5'(\zeta))$. Multiply these polynomials by the same binary form of degree $d-d_0$ with the roots at $\zeta_1, ..., \zeta_r$ of multiplicity $d_1, ..., d_r$ respectively (it is encoded by the polynomial $x(\zeta) = (\zeta-\zeta_1)^{d_1} ... (\zeta-\zeta_r)^{d_r}$ with the obvious convention about the roots at $\zeta = \infty$). Then the polynomials $(x_1(\zeta), ..., x_5(\zeta)) := (x(\zeta)x_1'(\zeta), ..., x(\zeta)x_5'(\zeta))$ represent the image $\mu(f)$ in the space LX_d.

Obviously, the map μ is equivariant with respect to the natural actions of the automorphism group $PSL_2(\mathbb{C}) \times PSL_5(\mathbb{C})$ of $\mathbb{C}P^1 \times X$ on $GX_{0,d}$ and $LX_d = Proj(H^0(\mathbb{C}P^1, \mathcal{O}_{\mathbb{C}P^1}(d)) \otimes H^0(X, \mathcal{O}_X(1))^*)$ and is therefore independent on the choice of the coordinate systems $(x_1 : ... : x_5)$ and ζ on X and $\mathbb{C}P^1$. In order to check that μ is regular let us assume for the moment that the map f is transverse to the five coordinate hyperplanes $\mathbb{C}P^1 \times x_i^{-1}(0)$, to $\infty \times X$, and that the intersections of $f(\Sigma)$ with the coordinate hyperplanes are away from $\infty \times X$. Then these intersections are all simple (even if f is a multiple cover, they are simple distinct non-singular points on Σ), their projections to $\mathbb{C}P^1$ are away from ∞ and determine the roots of the polynomials $x_1(\zeta),...,x_5(\zeta)$. The projective coordinates of the intersection point $f(\Sigma)\cap(\infty\times X)$ determine the top coefficients of the polynomials $x_1(\zeta),...,x_5(\zeta)$ uniquely up to a non-zero scalar factor (we can fix it by requiring that the top coefficient of $x_1(\zeta)$ equals 1). Now, consider the point $[f]$ represented by the map f in the moduli orbifold $GX_{0,d}$ and a local non-singular chart near this point. Due to the simplicity of the intersection points of $f(\Sigma)$ with the coordinate hyperplanes and with $\infty\times X$ the roots and top coefficients of the polynomials $x_i(\zeta)$ are regular functions of the map in a sufficiently small neighborhood of $[f]$ (if this is not obvious yet, consider the universal stable map $\mathrm{ev}: GX_{1,d}\to\mathbb{C}P^1\times X$ and describe the intersection points as local sections of the forgetting map $\mathrm{ft}: GX_{1,d}\to GX_{0,d}$). Thus the map μ is regular at $[f]$ since top coefficients and roots uniquely determine the polynomials.

Choosing the coordinate systems on $\mathbb{C}P^1$ and X in general position to a given f we conclude that μ is regular everywhere.

The map μ allows one to compare the equivariant Euler classes of LV_d and GV_d. Let $-p$ denote the equivariant Chern class of the Hopf bundle over LX_d pulled-back to $GX_{0,d}$. Introduce the formal series

$$G(q,z) := \sum_{d=0}^{\infty} q^d \int_{GX_{0,d}} e^{pz} Euler(GV_d).$$

Theorem B

$$G(q,z) = \langle J(qe^{z\hbar},\hbar^{-1}), J(q,-\hbar^{-1})\rangle\ ,$$

where the formal vector-function J with values in the cohomology algebra $\mathbb{Q}[P]/(P^4)$ is defined by

$$J(q,\hbar^{-1}) := e^{(P\ln q)/\hbar}[1+\hbar^{-1}\sum_{d=1}^{\infty} q^d\, \mathrm{ev}_*(\frac{Euler(V_d)}{\hbar - c})], \tag{3}$$

c denotes the Chern class of the line orbibundle over $X_{1,d}$ formed by cotangent lines to the curves at the marked point, the map $\mathrm{ev}: X_{1,d}\to X$ *is defined by evaluation of stable maps at the marked point, and the push forward is well-defined by*

$$\langle\phi, \mathrm{ev}_*\Psi\rangle = \int_{\mathbb{C}P^4} 5P\phi\, \mathrm{ev}_*\Psi := \int_{X_{1,d}} (\mathrm{ev}^*\phi)\Psi.$$

Proof (see [9], Section 6). It consists in application of fixed point localization in equivariant cohomology. A fixed point of the S^1-action on $GX_{0,d}$ is represented by a stable map $\Sigma \to GX$ which consists of the graph of a constant map $\mathbb{C}P^1 \to X$ with two "vertical " curves of degrees $d-d'$ and d' mapped to the slices $\zeta = 0$ and $\zeta = \infty$ and attached to the graph. Thus the fixed point components are given by the diagonal constraint at the marked points in the Cartesian product $X^{(0)}_{1,d-d'} \times X^{(\infty)}_{1,d'}$, $d' = 1, ..., d-1$, (and are isomorphic to $X^{(0)}_{1,d}$ and $X^{(\infty)}_{1,d}$ for $d' = 0, d$). The normal bundle to the fixed point component has the equivariant Euler class $(-\hbar)(-\hbar - c^{(0)})(\hbar)(\hbar - c^{(\infty)})$ (unless $d' = 0$ or d in which case a half of the product should be taken). The Euler classes occur in the denominator of the localization formula. The bundle GV_d restricted to the fixed point set splits into $V'_{d-d'} \oplus V'_{d'} \oplus \mathrm{ev}^* \mathcal{O}(5)$ where $'$ symbolizes that the subspace in $H^0(\Sigma, f^*\mathcal{O}(5))$ consisting of sections vanishing at the marked point is taken. Due to the multiplicative property of Euler classes, $Euler(GV_d)$ factors correspondingly. Since the map μ sends the fixed point component in GX_d to the corresponding fixed point component in LX_d, the class p localizes to $\mathrm{ev}^*(P) + d'\hbar$. It remains only to rearrange the summation over d and d' as the double sum over d' and $d'' = d - d'$. $\square$

The divisor equation. It is useful to figure out the place of the series J in the axiomatic structure of Gromov – Witten theory (see for instance Dubrovin's book [7]). Genus 0 Gromov – Witten invariants of a Kähler manifold Y define on $H^*(Y)$ the structure of a Frobenius manifold. It basically consists of the quantum cup-product in each tangent space and a family of flat connections $\nabla_\hbar$ on the tangent bundle defined by the quantum multiplication operators and depending on a parameter which we denote $\hbar^{-1}$. Structural constants of the quantum cup-product are third partial derivatives $F_{\alpha\beta\gamma}$ of the *potential*

$$F(T) = \sum_{n,d} \frac{q^d}{n!} (T, ..., T)_{n,d} \ ,$$

where T is a general cohomology class of Y,

$$(T, ..., T)_{n,d} := \int_{[Y_{n,d}]^{virt}} \mathrm{ev}_1^*(T) ... \mathrm{ev}_n^*(T),$$

and the “foreign” formal parameter q is introduced in order to provide convergence of the d-sum in l-adic topology at least. For instance, the numbers n_d for quintic threefolds are encoded by the *Yukawa coupling*

$$F_{ttt} = \sum_{n,d} \frac{q^d}{n!}(P,P,P,Pt,...,Pt)_{n+3,d}$$

which is obtained by differentiation of the potential restricted to the 2-nd cohomology space of the quintic and, as we will see soon, coincides with the formal Fourier series

$$K(qe^t) := \sum_{d=0}^{\infty}(P,P,P)_{3,d}q^d e^{dt} = 5 + \sum_{d=1}^{\infty}\frac{n_d d^3 q^d e^{dt}}{1-q^d e^{dt}}.$$

The vector fields on $H^*(Y)$ flat with respect to the connections $\nabla_\hbar$ also represent some Gromov – Witten invariants. Namely, the following matrix is the fundamental solution to the system of linear differential equations $\nabla_\hbar S = 0$:

$$S_{\psi\phi}(T,\hbar^{-1}) := \langle\psi,\phi\rangle + \sum_{n,d}\frac{q^d}{n!}(\psi,T,...,T,\frac{\phi}{\hbar - c})_{n+2,d}\ , \tag{4}$$

where $\phi, \psi \in H^*(Y)$, and c is the Chern class of the universal cotangent line at the last marked point (as specified by the position of this class in the correlator).

Of course, all the correlators for quintic threefolds Y are defined by means of intersection theory in $X_{n,d}$ with the Euler classes of the bundles V_d pulled-back from $X_{0,d}$ by forgetting maps.

Theorem C.

$$\langle J(qe^t,\hbar^{-1}), e^{-(P\ln q)/\hbar}\phi\rangle = \langle 1,\phi\rangle + \sum_{n,d}\frac{q^d}{n!}(1,Pt,...,Pt,\frac{\phi}{\hbar - c})_{n+2,d} \tag{5}$$

Proof ([9], Section 6). It consists in application of the *string equation* followed by an iterative application of the *divisor equation.* The string and divisor equations read respectively:

$$(1,T,...,T,\frac{\phi}{\hbar - c})_{n+2,d} = \hbar^{-1}(T,...,T,\frac{\phi}{\hbar - c})_{n+1,d}, \tag{6}$$

$$(P,T,...,T,\frac{\phi}{\hbar-c})_{n+2,d}=d\ (T,...,T,\frac{\phi}{\hbar-c})_{n+1,d}+\hbar^{-1}(T,...,T,\frac{P\phi}{\hbar-c})_{n+1,d}, \tag{7}$$

where one should require $n\geq 2$ in the case if $d=0$. The equations allow to push the classes $\mathrm{ev}_i^*(1)$ and $\mathrm{ev}_i^*(P)$ of degree ≤ 2 from $X_{2+n,d}$ to $X_{1,d}$ along the forgetting maps $X_{n+2,d}\to X_{n+1,d}\to ...\to X_{1,d}$. This transforms the sum of d-terms in the series (5) to $\hbar^{-1}(\ \phi\exp(t(P+d\hbar)/\hbar)/(\hbar-c)\ )_{1,d}$ for $d>0$ (and to $\langle\exp(tP/\hbar),\phi\rangle$ for $d=0$ since $X_{3,0}=X$). $\square$

A similar application of the divisor equation to the Yukawa coupling shows that $F_{ttt}(tP)=K(qe^t)$. On the other hand, applying the string equation to the elements $S_{1,P}(tP,\hbar^{-1})$ of the matrix (4) we extract the Yukawa coupling as the coefficient at $\hbar^{-2}$ in the second derivative in t. The theorem therefore implies that

$$K(e^t)=\langle\hbar^2\frac{d^2}{dt^2}J(e^t,\hbar^{-1}),P\rangle. \tag{8}$$

Thus in order to prove the quintic formula it suffices to identify J and I up to suitable mirror transformations.

We complete this subsection by including a proof of the string and divisor equations. The argument seems to be standard in Deligne – Mumford theory.

The maps $\mathrm{ft}: X_{k+1,d}\to X_k$ are defined [15, 3] by forgetting the first marked point ε_0 of a stable map $f:(\Sigma,\varepsilon_0,...,\varepsilon_k)\to X$ and producing a new stable map $\tilde{f}:(\tilde{\Sigma},\varepsilon_1,...,\varepsilon_k)\to X$ by contracting those components of Σ which have become unstable. The fiber of the forgetting map is canonically identified with the quotient of the curve $(\tilde{\Sigma},\varepsilon_1,...,\varepsilon_k)$ by the finite automorphism group of the map $\tilde{f}$. In particular the map ft has k canonical sections ε_i defined by the marked points in the fibers and together with the evaluation map $\mathrm{ev}_0: X_{k+1,d}\to X$ can be considered as the universal stable map with k marked points.

One derives the equations (6, 7) by comparing the Chern class c of the universal cotangent line bundle over $X_{1+k,d}$ with the pull-back $\tilde{c}$ of the corresponding class from $X_{k,d}$. The cotangent lines $T^*_{\varepsilon_k}\Sigma$ and $T^*_{\varepsilon_k}\tilde{\Sigma}$ are canonically identified *unless* ε_k is situated on the same irreducible component Σ_0 of Σ as ε_0, the map f restricted to Σ_0 is constant, Σ_0 carries no other marked points and contains only one singular point of the curve Σ. Since $\Sigma_0\simeq\mathbb{C}P^1$ carries

3 special points of different nature, the cotangent line $T^*_{\varepsilon_k}\Sigma$ is canonically trivialized in this case. In terms of the universal cotangent line bundles l_k and $\tilde{l}_k = \mathrm{ft}^*\, l_k$ over $X_{k,d}$ with Chern classes c and $\tilde{c}$ this means that $\tilde{l}_k^* \otimes l_k$ has a canonical section non-vanishing outside the divisor $\varepsilon_k(X_{k,d})$ defined by the universal marked point $\varepsilon_k : X_{k,d} \to X_{k+1,d}$, and that l_k restricted to this divisor is trivial. Since the restriction of $\tilde{l}_k$ to the divisor coincides with the conormal bundle to this divisor by the very definition of the universal cotangent line $T^*_{\varepsilon_k}\tilde{\Sigma}$ on $X_{k,d}$, one concludes that the section has the 1-st order zero along the divisor. Thus $\delta = c - \tilde{c}$ is Poincare-dual in $X_{k+1,d}$ to the hypersurface $\varepsilon_k(X_{k,d})$, and $c\delta = 0$.

We have

$$(1, T, ..., T, \tilde{c}^m \phi)_{k+1,d} = 0,\ \ (P, T, ..., T, \tilde{c}^m\phi)_{k+1,d} = d(T, ..., T, c^m\phi)_{k,d}$$

since the integral of the classes 1 and P over degree d curves equal 0 and d respectively. Finally, replacing $\tilde{c}$ by $c = \tilde{c} + \delta$ yields the extra-terms $(T, ..., T, c^{m-1}\phi)_{k,d}$ and $(T, ..., T, c^{m-1}P\phi)_{k,d}$. This implies the string and divisor equations since $1/(\hbar - c)$ is the eigenfunction of the operation $c^m \mapsto c^{m-1}$ with the eigenvalue $1/\hbar$. $\square$.

Torus action. A link between I and J can be established via a recursion relation satisfied by their equivariant perturbations I^{eq} and J^{eq}.

Consider the standard action of the 4-dimensional torus G on $X = \mathbb{C}P^4$. The equivariant cohomology algebra $H^*_G(X)$ is generated over the coefficient ring $\mathbb{Q}[\lambda] = H^*(BG)$ by the equivariant Chern class $-P$ of the Hopf line bundle and satisfies the relation

$$(P - \lambda_1)(P - \lambda_2)(P - \lambda_3)(P - \lambda_4)(P - \lambda_5) = 0$$

(where we assume that $\lambda_1 + ... + \lambda_5 = 0$). Evaluation of an equivariant cohomology class on the invariant fundamental class can be computed via fixed point localization:

$$\int_{[X]} \phi(P, \lambda) = \sum_{\alpha=1}^{5} \frac{\phi(\lambda_\alpha, \lambda)}{e_\alpha} = \frac{1}{2\pi i} \oint \frac{\phi(P, \lambda)dP}{(P - \lambda_1)...(P - \lambda_5)}$$

where $e_\alpha = \Pi_{\beta \neq \alpha}(\lambda_\alpha - \lambda_\beta)$ is the equivariant Euler class of the tangent space to X at the fixed point r_α where P restricts to λ_α. The equivariant Chern class of the anti-canonical line bundle $\mathcal{O}_X(5)$ equals $5P$, and we will denote $\langle \phi, \psi \rangle$ the equivariant intersection pairing on $H^*_G(X)$ with this Chern class:

$$\langle \phi, \psi \rangle := \int_{[X]} 5P\phi\psi.$$

The torus G acts on the moduli orbifolds $X_{n,d}$ which allows one to introduce the equivariant Gromov – Witten correlators (such as $(T, ..., T)_{n,d}$ and $(\psi, T, ..., T, \phi c^k)_{n+2,d}$ where c and $\phi, \psi, T \in H^*_G(X)$ are equivariant cohomology classes. The correlators take values in $\mathbb{Q}[\lambda]$ and turn into corresponding non-equivariant Gromov – Witten invariants when specialized to $\lambda = 0$.

Similarly, one can use $S^1 \times G$-equivariant intersection theory in the spaces LX_d and $GX_{0,d}$ and carry over Theorems $A(a), B, C$ to the equivariant setting. The proofs are identical to those given above, but the equivariant counterpart of the hypergeometric series I is defined now by the series

$$I^{eq} := e^{(P \ln q)/\hbar} \sum_{d=0}^{\infty} q^d \frac{\Pi_{m=1}^{5d}(5P + m\hbar)}{\Pi_{m=1}^{d} \Pi_{\alpha=1}^{5}(P - \lambda_\alpha + m\hbar)}, \tag{9}$$

with coefficients in the *equivariant* cohomology algebra of X generated by P (and represents a fundamental solution to some 5-th order ODE, which is however irrelevant for our goal in this paper).

In order to describe the recursion relation satisfied by the hypergeometric series I^{eq} let us strip off the factor $\exp(P \ln q)/\hbar$, denote the remaining series by $Z^{(hg)}$ and denote by $Z^{(hg)}_\alpha$ its fixed point localizations:

$$Z^{(hg)}_\alpha(q, \hbar^{-1}, \lambda) = \sum_{d=0}^{\infty} \frac{q^d}{d!\hbar^d} \frac{\Pi_{m=1}^{5d}(5\lambda_\alpha + m\hbar)}{\Pi_{m=1}^{d} \Pi_{\beta \neq \alpha}(\lambda_\alpha - \lambda_\beta + m\hbar)} \ .$$

Coefficients of the formal q-series $Z^{(hg)}_\alpha$ are degree 0 rational functions in $\hbar$ with the first order pole at $\hbar = (\lambda_\beta - \lambda_\alpha)/m$, $\beta \neq \alpha$, $m = 1, ..., d$, and a high order pole at $\hbar = 0$. Rewriting the rational functions as sums of elementary fractions we arrive at the following recursion relation:

$$Z_\alpha(q, \hbar^{-1}, \lambda) = \tag{10}$$

$$1 + \sum_{d=1}^{\infty} q^d \frac{R_{\alpha,d}(\hbar, \lambda)}{\hbar^d} + \sum_{\beta \neq \alpha} \sum_{m=1}^{\infty} C^\beta_\alpha(m) \frac{q^m}{\lambda_\alpha - \lambda_\beta + m\hbar} Z_\beta(q, \frac{m}{\lambda_\beta - \lambda_\alpha}, \lambda),$$

where $R_{\alpha,d}$ are some degree $\leq d$ polynomials in $\hbar$ with coefficients rational in λ, and $C_\alpha^\beta(m)$ are rational functions in λ which we will call the *recursion coefficients*. Since $Z_\beta^{(hg)} \equiv 1(\mod q)$, the coefficient $C_\alpha^\beta(d)$ can be read off the q^d-term of the series $Z_\alpha^{(hg)}$ as the residue at the pole $\hbar = (\lambda_\beta - \lambda_\alpha)/d$.

We will refer to the sequence of polynomials $R_{\alpha,d}$ as the *initial condition*: given such an initial condition, the recursion relation allows to recover the solution $\{Z_\alpha, \alpha = 1, ..., 5\}$ unambiguously.

Now, starting with the Gromov – Witten invariant J^{eq}, introduce the vector q-series $Z^{(GW)}(q, \hbar^{-1}, \lambda)$ with coefficients in the algebra $H_G^*(X, \mathbb{Q}(\hbar))$ by stripping off the factor $\exp(P \ln q)/\hbar$, and denote by $Z_\alpha^{(GW)}$ the localization of $Z^{(GW)}$ at the fixed r_α.

Theorem D ([9], Section 11). *The series $\{Z_\alpha^{(GW)}, \alpha = 1, ..., 5\}$ satisfy the recursion relation (10) with the same recursion coefficients $C_\alpha^\beta(d)$ (and with another initial condition).*

Proof ([9], Sections $9, 11$). It is based on localization to fixed points of the torus G action on the moduli orbifolds $X_{1,d}$. Consider a stable map $f : (\Sigma, \varepsilon) \to X$ representing such a fixed point. The combinatorial structure of the curve Σ is described by a tree of irreducible components (isomorphic to $\mathbb{C}P^1$ each). Some components are mapped onto the straight lines in $X = \mathbb{C}P^4$ connecting the fixed points $r_\alpha, \alpha = 1, ..., 5$, of the torus action, and the map is a multiple cover $\zeta \mapsto \zeta^m$ in suitable affine coordinates on the source and target $\mathbb{C}P^1$, so that $\zeta = 0, \infty$ are mapped to the fixed points. The remaining irreducible components of Σ are mapped to the fixed points in X. The marked point ε must be mapped to one of the fixed points r_α.

The fixed point in $X_{1,d}$ represented by f does not contribute to $Z_\alpha^{(GW)}$ via localization formulas *unless* $f(\varepsilon) = r_\alpha$.

Suppose that ε is situated in an irreducible component of Σ mapped to r_α. Consider the whole connected component of the fixed point set $X_{1,d}^G$ in $X_{1,d}$ which contains the equivalence class $[f]$. We will show that this connected component contributes to $Z_\alpha^{(GW)}$ by a polynomial in $\hbar^{-1}$. Indeed, the component can be described as the (quotient by a finite group of the) product of some Deligne – Mumford spaces $\bar{\mathcal{M}}_{0,k}$ (why? — see [15] where the fixed point set is described). The universal cotangent line orbibundle over $X_{1,d}$ restricted to the connected component of $X_{1,d}^G$ coincides with the universal cotangent line at one of the marked points over one of the factors

$\bar{\mathcal{M}}_{0,k}$. Thus its Chern class c is nilpotent on this component, and the geometrical series $(\hbar - c)^{-1}$ reduces to a finite sum of terms $c^l/\hbar^{l+1}$. Notice that $l \leq \dim \bar{\mathcal{M}}_{0,k} = k - 3$ is bounded by the total degree of the map f.

The fixed point localization terms just discussed form the initial condition in (10). We will show that contributions of all other fixed points can be arranged as the recursive part of (10). The idea is to cut off the component of the curve $(\Sigma, \varepsilon) \to X$ carrying the marked point and to observe that the rest of the curve represents a torus-invariant curve of smaller degree.

In greater detail, suppose that ε is situated at $\zeta = 0$ on a multiple cover $\zeta \mapsto \zeta^m$ of the line connecting r_α with r_β. Then the universal cotangent line orbibundle restricted to the connected component of $X^G_{1,d}$ is topologically trivial (since $(T^*_\varepsilon \Sigma)^{\otimes m}$ coincides in this case with the cotangent space $T^*_{r_\alpha}\mathbb{C}P^1$ to the line joining r_α and r_β), but it carries a nontrivial infinitesimal action of G given by the character $(\lambda_\alpha - \lambda_\beta)/m$. Thus the localization of $(\hbar - c)^{-1}$ at this fixed point component yields the simple fraction $mq^m(m\hbar + \lambda_\beta - \lambda_\alpha)^{-1}$. The factor m is eventually compensated by the order of the automorphism group of the map $\zeta \to \zeta^m$ which occurs in the denominator of localization formulas on orbifolds. The weight q^m counts the degree of this map as a curve in X.

The whole contribution of the fixed point component to $Z^{(GW)}_\alpha$ via localization formulas includes two more factors. Each of them takes in account the equivariant Euler classes of the orbibundle V'_d and of the normal orbibundle to $X^G_{1,d}$ which occurs in the denominator of localization formulas.

The first factor corresponds to the irreducible component $C = \mathbb{C}P^1$ of Σ carrying the marked point, and the second one — corresponds to the remaining part $\tilde{\Sigma}$ of the curve Σ. The map f restricted to $\tilde{\Sigma}$ has degree $d - m$ and represents a point in the space $X^G_{1,d-m}$. The fiber $H^0(\Sigma, f^*V')$ of V'_d contains the subspace of sections vanishing on C, which coincides with the fiber of V'_{d-m}. The normal spaces to $X^G_{1,d}$ split similarly into parts corresponding to C and $\tilde{\Sigma}$. The intersection point $\zeta = \infty$ of C with $\tilde{\Sigma}$ plays the role of the marked point $\tilde{\varepsilon}$ in $\tilde{\Sigma}$. The deformation of f corresponding to smoothening of the curve Σ at the double point $\tilde{\varepsilon}$ is represented in the tangent space to $X_{1,d}$ by the line $T_{\tilde{\varepsilon}}C \otimes T_{\tilde{\varepsilon}}\tilde{\Sigma}$ and contributes the factor $(\lambda_\beta - \lambda_\alpha)m^{-1} - \tilde{c}$ to the denominator of the localization formula. Thus the contribution of $\tilde{\Sigma}$ is correctly accounted by the factor $Z^{(GW)}_\beta(q, m/(\lambda_\beta - \lambda_\alpha), \lambda)$ in the recursion relation (10).

The remaining factor in the localization formula is a rational function of λ and can be computed explicitly as the ratio of two Euler classes in the case $m = d$ when $\tilde{\Sigma}$ is a point (in the example of quintics we are studying, the factor has actually been computed in [15]). It turns out to coincide with the recursion coefficient $C_\alpha^\beta(m)$. We recommend the reader to carry out this computation (it amounts to analyzing the torus action on spaces of holomorphic sections of $\mathcal{O}(5)$ and T_X lifted to the multiple cover $\zeta \mapsto \zeta^m$ of the line joining r_α and r_β) or at least to look at some details of this computation in [15]. $\square$

A plausible argument in [9, 11] intended to explain the "miraculous" coincidence of the recursion coefficients has been formalized in [17].

Polynomiality. The recursion relation (10) has much more solutions with various initial conditions than the mirror transformations can handle. However, according to Theorem A and Theorem B, the solutions $Z^{(hg)}$ and $Z^{(GW)}$ have the following polynomiality property.

Let us call a solution $Z(q, \hbar^{-1}, \lambda)$ to the recursion relation (10) *polynomial* if the formal (q, z)-series

$$\langle Z(qe^{\hbar z}, \hbar^{-1}, \lambda), e^{Pz}\ Z(q, -\hbar^{-1}, \lambda)\rangle \tag{11}$$

has coefficients polynomial in $\hbar$.

The solution $Z^{(GW)}$, by the very definition (3), satisfies also the asymptotical condition

$$Z(q, \hbar^{-1}, \lambda) = 1 + o(\hbar^{-1}). \tag{12}$$

Theorem E ([9], Proposition 11.5). *A polynomial solution to the recursion relation (10) satisfying the asymptotical condition (12) is unique.*

Proof: perturbation theory. Let Z be a polynomial solution and let $\delta R = R_d - R_d^{(GW)}$ denote the discrepancy in the initial conditions for Z and $Z^{(GW)}$ with minimal $d > 0$. Then Z and $Z^{(GW)}$ coincide modulo q^d due to the recursion relation. The polynomiality property for Z and $Z^{(GW)}$ modulo q^{d+1} translates into regularity at $\hbar = 0$ of

$$\langle \delta R(\hbar)\hbar^{-d}, e^{(P+d\hbar)z}\rangle + \langle \delta R(-\hbar)(-\hbar)^{-d}, e^{Pz}\rangle.$$

Localizations of this intersection index to fixed points in X are — for each power of $\hbar^{-1}$ — finite sums of monomials $z^l \exp(\lambda_\alpha z)$, $l = 0, 1, 2, \ldots$, $\alpha =$

$1,...,5$. Linear independence of such monomials for generic λ implies that the localizations

$$\delta R_\alpha(\hbar)\hbar^{-d}e^{\hbar z}+\delta R_\alpha(-\hbar)(-h)^{-d},$$

where $R_\alpha(\hbar)$ are some polynomials in $\hbar$ of degree $\leq d$, must be regular at $\hbar=0$ on their own.

Consider first $(A\hbar^{-2}+B\hbar^{-3})\exp(\hbar z)+(A\hbar^{-2}-B\hbar^{-3})$ modulo z^2. The regularity condition at $\hbar=0$ implies $A=0$ and then $B=0$. Applying this argument inductively we conclude that $\delta R_\alpha(\hbar)\hbar^{-d}=A_\alpha+B_\alpha\hbar^{-1}$ where A_α, B_α do not depend on $\hbar$.

Assuming now that Z also satisfies the asymptotical condition (12) we find $\delta R=0$. $\square$

Mirror transformations. The hypergeometric series I^{eq} has the asymptotical expansion

$$I^{eq}=e^{(P\ln q)/\hbar}(f_0(q)+f_1(q)\frac{P}{\hbar}+o(\hbar^{-1})),$$

where the series

$$f_0=\sum_{d=0}q^d\frac{(5d)!}{(d!)^5},$$

$$f_1=\sum_{d=1}q^d\frac{(5d)!}{(d!)^5}(\sum_{m=d+1}^{5d}\frac{5}{m}),$$

are found from (9) (remember that $\lambda_1+...+\lambda_5=0$).

The *mirror transformations*, namely the division of I^{eq} by f_0 followed by the change of variable $\ln q\mapsto\ln q + f_1(q)/f_0(q)$, transform I^{eq} to a new vector-function with the same asymptotical behavior as

$$J^{eq}=e^{(P\ln q)/\hbar}(1+o(\hbar^{-1})).$$

Therefore the following theorem guarantees that the transformed series coincides with J^{eq}. Passing to the non-equivariant limit $\lambda=0$ we conclude that the same mirror transformations take I into J.

Thus the Yukawa coupling (8) is indeed extracted from the fundamental solution I to the differential equation (1) by the procedure conjectured in [6].

Theorem F ([9], Propositions 11.3, 11.6). *The mirror transformations take polynomial solutions of the recursion relation (10) to polynomial solutions of the same recursion relation.*

Proof: straightforward (see [9]). The division operation does not change the form of the recursion relation and also preserves the polynomiality property since the extra factor $f_0(q\exp(\hbar z))f_0(q)$ does not produce negative powers of $\hbar$ in the (q,z)-series (11). The change of the variables $\ln q \mapsto \ln q + g(q)$ transforms z in this series into $z + [g(q\exp(\hbar z)) - g(q)]/\hbar$. At $\hbar = 0$ the difference vanishes. It is therefore divisible by $\hbar$, and thus the change of variables preserves the polynomiality property too.

When applied to the recursion relation (10) literally, the change of variables modifies it to a new recursion relation. In the new form the elementary fraction

$$\frac{q^m}{m\hbar + \lambda_\alpha - \lambda_\beta} = \frac{q^m}{\hbar(\lambda_\alpha - \lambda_\beta)} \quad \frac{1}{\hbar^{-1} + m/(\lambda_\alpha - \lambda_\beta)}$$

occurs with the extra factor

$$\delta := \exp[mg(q) + \lambda_\alpha g(q)\hbar^{-1} - \lambda_\beta g(q)\frac{m}{\lambda_\beta - \lambda_\alpha}]$$

$$= \exp[\lambda_\alpha g(\hbar^{-1} + m/(\lambda_a - \lambda_\beta))].$$

Thus $\delta - 1$ is divisible by $\hbar^{-1} + m/(\lambda_\alpha - \lambda_\beta)$. Since $g(0) = 0$, the result of this division is a q-series with coefficients polynomial in $\hbar^{-1}$ at each power of q. Thus the transformation affects only the initial condition and takes a solution of the recursion relation into another solution. $\square$

Updates.

Definitions. The definition of virtual numbers n_d in terms of Euler classes of the orbibundles V_d over $X_{0,d}$ should be considered as tentative and has been replaced by a more universal construction, due to J. Li & G. Tian [16], of virtual fundamental classes $[Y_{0,d}]$ defined in intrinsic terms of the quintic 3-fold Y rather than in terms of the embedding $i : Y \subset X$. Thus in order to place the above proof of the quintic formula into the framework of contemporary definitions one needs to check that $i_*[Y_{0,d}]$ in $H_*(X_{0,d})$ equals the cap-product of the virtual fundamental class $[X_{0,d}]$ with $Euler(V_d)$. This is easy and can be done as follows.

Consider first a model problem studied in [8]: given a holomorphic section $s : B \to E$ of a vector bundle $E \to B$ over a compact complex manifold, construct a cycle in the zero locus $Z = s^{-1}(0)$ of the section which is Poincare-dual to the Euler class of the bundle. The model problem is solved by the normal cone construction: the normal cone $C \subset E|_Z$ to the zero locus has pure dimension $\dim B$, and the homological intersection in $E|_Z$ of the fundamental cycle $[C]$ with that of the zero section represents in $H_*(Z)$ the required Euler class. The construction can be adjusted to the orbifold/orbibundle situation.

The virtual fundamental cycle construction in [16] is based on the observation that the normal cone $C \subset E|_Z$ is intrinsic with respect to the scheme structure of Z and the *tangent-obstruction* complex $ds : T_B|_Z \to E|_Z$ of vector bundles over Z defined by the differential of the section. The kernel of $ds|_{z\in Z}$ is the algebraic tangent space $T_z Z$. In the case when Z is a moduli space of stable maps to Y, J. Li & G. Tian exhibit a tangent-obstruction complex of orbibundles $T \to E$ with this property and by this define the intrinsic normal orbi-cone $C \subset E$ and the virtual fundamental class $[Z] := [C] \cap [\textit{zero section}]$ in $H_*(Z, \mathbb{Q})$.

In our situation $Z = Y_{0,d}$ is given in the orbifold $X_{0,d}$ by a section s of $V_d : E_{0,d} \to X_{0,d}$. Using the exact sequence $0 \to T_Y \to T_X|_Y \to N_Y \to 0$ where N_Y is the normal bundle to Y in X we obtain the exact sequence

$$0 \to H^0(\Sigma, f^*T_Y) \to H^0(\Sigma, f^*T_X) \to H^0(\Sigma, f^*N_Y) \to H^1(\Sigma, f^*T_Y) \to 0$$

for each stable map $f : \Sigma \to Y$. Since the fibers of $V_d|_Z$ coincide with $H^0(\Sigma, f^*N_Y)$, this implies (via the description [15, 16] of the algebraic tangent space $T_{[f]}Y_{0,d}$ in cohomological terms of deformation theory) that the complex $ds : T_{X_{0,d}}|_Z \to E_{0,d}|_Z$ can be taken on the role of the tangent-obstruction complex in the definition of the virtual fundamental cycle $[Y_{0,d}]$. Thus this cycle represents in $H_*(X_{0,d})$ the Euler class of V_d.

While this obvious argument shows that the GW-invariant of Y in question can be computed in terms of the GW-theory for the convex bundle $V : E \to X$, some other GW-invariants of Y can not be interpreted in terms of the bundle. In order to distinguish the GW-theory of the bundle from the GW-theory of Y we, following A. Schwarz, refer in [11] to the first one as the GW-theory of the *supermanifold* ΠE.

The map μ. Applications of our approach to complete intersections in toric manifolds more general than projective spaces showed that some steps in the above proof are redundant. In the remaining part of the text we discuss several such steps which can be simplified or avoided. The first of them is our use of the map $\mu : GX_{0,d} \to LX_d$.

The map was used in order to define the equivariant class p on $GX_{0,d}$ as a pill-back of the corresponding class on LX_d and thus assure the polynomiality property of the GW-invariant J^{eq}. Consider instead the following $S^1 \times G$-equivariant GW-invariant of $\mathbb{C}P^1 \times X$:

$$\sum_{d=0}^{\infty} \sum_{n=0}^{\infty} \frac{q^d}{n!} [T, ..., T]_{n,d}, \tag{13}$$

where $T = z(p \otimes P)$, p is the generator of the S^1-equivariant cohomology of $\mathbb{C}P^1$ satisfying $p(p - \hbar) = 0$, P is the generator of the G-equivariant algebra of X, and $[...]_{n,d}$ is the equivariant GW-invariant defined by integration over $GX_{n,d}$ against the Euler class of $GV_{n,d}$. The series (13) is defined without fixed point localization and thus is a (q, z)-series with coefficients polynomial in $\hbar$ and λ. On the other hand, applying localization to fixed points of S^1-action on $GX_{n,d}$ as in the proof of Theorem B (notice that p localizes to 0 at $\zeta = 0$ and to $\hbar$ at $\zeta = \infty$) and then using the divisor equation for zP as in the proof of Theorem C we will find that the series (13) coincides with $\langle J^{eq}(q \exp \hbar z, \hbar^{-1}), J^{eq}(q, -\hbar^{-1}) \rangle$. This argument was mentioned in [11] and was used in [12].

Theorem F. The invariance of the recursion relation (10) under mirror transformations can be deduced from the string and divisor equations. Namely, consider the G-equivariant GW-invariant $\mathcal{J}^{eq}$ defined by integration over $X_{n+2,d}$ against the Euler classes of $V_{n+2,d}$:

$$\langle \mathcal{J}, \phi \rangle = \langle 1, \phi \rangle + \sum_{(n,d) \neq (0,0)} \frac{q^d}{n!} (1, T, ..., T, \frac{\phi e^{P \ln q / \hbar}}{\hbar - c})_{n+2,d} \tag{14}$$

with $T = a(q) + b(q)P$ where a, b are power q-series vanishing at $q = 0$. One can derive a recursion relation for $\mathcal{J}^{eq}$ in exactly the same way as we derived the recursion relation for J^{eq} in Theorem D. The recursion coefficients will be the same as in Theorem D, but the initial condition will depend now on a and b. On the other hand the string and divisor equations show that

$$\mathcal{J}^{eq} = e^{a(q)} J^{eq}(q e^{b(q)}, \hbar^{-1})$$

and is therefore a result of a mirror transformation applied to J^{eq}. Also, using the argument from the previous subsection with suitable function of a

and b on the role of T, we can make sure that $\mathcal{J}^{eq}$ must *a priori* satisfy the polynomiality condition as well.

Mirror transformations. As it was shown in [9] (Section 12) and [12] (Section 5), there is a "non-linear Serre duality" equivalence between genus 0 equivariant GW-theory for a convex supermanifold ΠE and such a theory for the non-compact total space E^* of the concave dual bundle $V^* : E^* \to X$. Namely, their genus 0 GW-invariants differ by a change of variables which can be explicitly described in terms of GW-invariants of either of them (see [12], Section 5).

On the other hand, the mirror formulas can be generalized, as it was shown in [17] and [12] (Section 4), to include genus 0 equivariant GW-invariants of concave bundles E^*. The proof is completely parallel to the one given above. However, in the case if the bundle V^* is the direct sum of at least two line bundles, the GW-invariant $J^{eq}_{E^*}$ for E^* is *equal* to the corresponding hypergeometric series $I^{eq}_{E^*}$ which in this case happens to satisfy the asymptotical condition of the uniqueness Theorem E. With this observation, a proof of the quintic formula looks as follows (see Section 5 in [12]).

Describe quintic 3-folds by two equations in $\mathbb{C}P^5$ of degree 1 and 5. For the concave bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-5)$ over $\mathbb{C}P^5$, prove the equality $J^{eq}_{E^*} = I^{eq}_{E^*}$ following the steps Theorem B — Theorem D — Theorem E as explained above. Now the mirror transformation between $J^{eq}_{\Pi E}$ and $I^{eq}_{\Pi E}$ emerge from the general formulas of "non-linear Serre duality".

Theorem D. The proof of mirror formulas given in [17] is based on some recursive property of the $S^1 \times G$-equivariant Euler classes of the bundles GV_d over $GX_{0,d}$ named *eulerity*. In fact the property can be easily deduced from our recursion relation (10) as it is done in [9] (Proposition 11.4 (b)). The inverse implication is also immediate. However, the proof of the eulerity property given in [17] is based on an argument similar to our proof of Theorem B (localization for the S^1-action) and thus completely eliminates the role of localization formulas for the G-action as a computational tool (and uses only the very fact of their existence).

With this observation, the reduction of the quintic formula to "non-linear Serre duality" theorem looks particularly short: prove eulerity property in the case of the concave bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-5)$ over $\mathbb{C}P^5$ following the argument in [17] and use a uniqueness theorem (parallel to Theorem E above) in order to identify $J^{eq}_{E^*}$ with $I^{eq}_{E^*}$.

We see that the only new, after M. Kontsevich's paper [15], geometrical construction which has survived so far through all variants of the proof of the quintic formula is the S^1-*equivariant* theory on the graph spaces $GX_{0,d}$ which originates from the loop space interpretation [10] (see also [21]) of Gromov – Witten invariants.

All other ingredients of the proof have somewhat combinatorial character and can be interchanged and simplified. This progress does not mean however that the mirror symmetry phenomenon has been adequately understood.

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