HOMOLOGICAL GEOMETRY AND MIRROR SYMMETRY

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0. A popular example. A homogeneous degree 5 polynomial equation in 5 variables determines a *quintic* 3-fold in $\mathbb{C}P^4$. Hodge numbers of a non-singular quintic are known to be: $h^{p,p} = 1, p = 0, 1, 2, 3$ (Kahler form and its powers), $h^{3,0} = h^{0,3} = 1$ (a quintic happens to bear a holomorphic volume form), $h^{2,1} = h^{1,2} = 101 = 126 - 25$ (it is the dimension of the space of all quintics modulo projective transformations, and $h^{2,1}$ is responsible here for infinitesimal variations of the complex structure) and all the other $h^{p,q} = 0$.

Consider the family of quintics $x_1 ... x_5 = \lambda^{1/5}(x_1^5 + ... + x_5^5)$ invariant to 5^4 multiplications of the variables by 5-th roots of unity. The quotient by these symmetries will generate singularities. Resolve the singularities. The result is known to be a family Y_λ of 3-folds with the table of Hodge numbers *mirror-symmetric* to that of the quintics X: $h^{p,q}(Y) = h^{3-p,q}(X)$.

Manifolds with mirror-symmetric Hodge tables are called *geometrical mirrors*. Discovered accidentally in a computer experiment, such mirror 3-folds very soon took their place in various string models of the 10-dimensional Universe. As it is clear now, so called Arnold's strange duality of exceptional singularities [1] was probably the first manifestation of mirror phenomena — for $K3$-surfaces.

Current interest to mirror manifolds is due to the so called *mirror conjecture* and its first applications to enumerative algebraic geometry. The idea is that along with the equality $h^{1,1}(X) = h^{2,1}(Y)$ of moduli numbers of Kahler structures on X and of complex structures on Y, whole symplectic topology on X is equivalent to complex geometry on Y, and vice versa.

This idea have led to a number of beautiful predictions (see for instance [6, 5]) in enumerative algebraic geometry, in particular – for numbers of rational curves of each degree on the quintics.

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1. Singularity theory. Given a complex manifold Y^n, a holomorphic volume form ω and a holomorphic function $f : Y \to \mathbb{C}$, one can study exponential integrals $I_\hbar = \int_\Gamma e^{f(y)/\hbar}\omega$, their asymptotics at $\hbar \to 0$ and their dependence on parameters.

Example. Let f be a weighted-homogeneous polynomial of $\deg f = 1$ on n complex variables $(y_1, ..., y_n)$ of some positive weights $\deg y_i = \alpha_i > 0$ with an isolated critical point $y = 0$ of multiplicity μ, a_1, ..., $a_\mu = 1$ — monomials representing a basis in the local algebra $H = \mathbb{C}[y]/(\partial f/\partial y)$, $f_\lambda = f + \lambda_1 a_1 + ... + \lambda_\mu a_\mu$ — a miniversal deformation of the critical point. The formal stationary phase approximation gives

$$I_i(\lambda) = \int e^{f_\lambda(y)} a_i(y) dy_1 \wedge ... \wedge dy_n \sim \hbar^{n/2} e^{f_\lambda(y_*)/\hbar} \frac{a_i(y_*)}{\sqrt{J_\lambda(y_*)}}$$

for each of μ critical points $y_*(\lambda)$ of f_λ, where $J_\lambda = \det(\partial^2 f_\lambda/\partial y^2)$. These asymptotics satisfy $\hbar\partial I_j/\partial\lambda_i \sim \sum_k c_{ij}^k(\lambda) I_k$ where c_{ij}^k are structural constants of the algebra $H_\lambda = \mathbb{C}[y]/(\partial f_\lambda/\partial y)$ of functions on the critical set: $a_i a_j = \sum c_{ij}^k a_k$ in H_λ. The cycles of integration can be described as real n-dimensional Morse-theoretic cycles of the function Ref and thus correspond to the critical points and represent classes in the asymptotical homology group $H_n(\mathbb{C}^n, Ref = -\infty)$. Then the residue paring

$$(a, b) = \sum_{y_*} \frac{a(y_*)b(y_*)}{J(y_*)} = \frac{1}{(2\pi i)^n} \int_{|\partial f/\partial y| = const} \frac{a(y)b(y) dy_1 \wedge ... \wedge dy_n}{\partial f_\lambda/\partial y_1 ... \partial f_\lambda/\partial y_n}$$

becomes an asymptotical intersection pairing between the cohomology for f and $-f$ and is known to extend without singularities to $\lambda = 0$.

Theorem[16] *These stationary phase asymptotics can be made exact by a suitable choice of the volume forms ω_λ instead of $dy_1 \wedge ... \wedge dy_n$ and in special coordinates $\tilde\lambda$ on the parameter space, instead of $(\lambda_1, ..., \lambda_\mu)$.*

In particular this means that the differential equations $\hbar\partial_i \vec{I} = [a_i] \cdot \vec{I}$ with $[a_i]$, $I(\tilde\lambda) \in H_{\tilde\lambda}$ form a family $\nabla_\hbar = \hbar d - \sum [a_i] d\tilde\lambda_i$ of connections *flat for all* $\hbar$. They are identified with the Gauss–Manin connections in the cohomological bundle. The residue pairing therefore literally coincides with the intersection pairing and induces (see [17]) on the parameter space a flat complex metric. The coordinates $\tilde\lambda$ are defined as flat coordinates of this metric.

In the contemporary language this theorem means that the integrals define on the parameter space the structure of a Frobenius manifold [7] and thus satisfy axioms of Topological Conformal Field Theory (TCFT) (*Landau – Ginzburg models* of TCFT).

One can at least try to extend this theorem based on deep properties of variations of Hodge structures to arbitrary families $(Y, f, \omega)_\lambda$. Consider a degenerate case where Y is a compact Kahler manifold. For this, Y should bear a holomorphic volume form $(h^{n,0} = 1)$. Then f is necessarily constant, and the exponential integrals turn into periods $\int \omega^{n,0}$ of the volume form. The periods depend on the complex structure on Y and satisfy some linear differential *Picard – Fuchs* equations (describing variation of pure Hodge structures). The algebra H_λ of the critical set Y should be replaced by its cohomology $H^n(Y)$. It is a separate problem whether one can derive from these 'massless' Landau – Ginzburg data complete models of TCFT (they are called B-models, after E.Witten), but in many cases one can construct flat coordinates on moduli spaces of complex structures. The family of quintic-mirrors Y_λ — is one of them.

2. Symplectic topology. Let X^n be a compact Kahler manifold. Given m cycles $A_1, ..., A_m \subset X$, an integral homology class $D \in H_2(X)$ and a configuration $(x_1, ..., x_m)$ of m points on $\mathbb{C}P^1$, one may ask: *how many holomorphic maps* $\varphi : \mathbb{C}P^1 \to X$ *with* $\varphi_*[\mathbb{C}P^1] = D$ *and* $\varphi(x_i) \in A_i$ *are there?* The answers (let us denote them $F_{m,D}[A]$), being properly understood as intersection indices in certain moduli spaces of holomorphic maps, turn out to depend only on homology classes of A_i and homotopy type of *almost* Kahler structure on X and provide symplectic invariants of X called *Gromov-Witten* invariants. They are not independent, and the universal identities for them can be interpreted as the associativity constraint of the *quantum cohomology* algebra $H_q^*(X)$ and compatibility of some linear PDEs.

Pick an integral basis $p_1, ..., p_k$ of symplectic classes in $H^2(X)$, denote $(D_1, ..., D_k)$ coordinates of D in the dual basis and put $(a_1|a_2|...|a_m) = \sum_D q^D F_{m,D}[A_1, ..., A_m]$ where A_i are Poincare dual to cohomology classes a_i.

Theorem [14, 15]. *Gromov - Witten invariants are well - defined at least if* $c_1(X) \geq 0$, $(a_1|a_2)$ *coincides with the Poincare pairing* (a_1, a_2) *on* $H_q^*(X) = H^*(X, \mathbb{C}[[q]])$, $(a_1|a_2|a_3)$ *are structural constants* $(a_1 * a_2, a_3)$ *of a skew - commutative associative multiplication* $*$ *on* $H_q^*(X)$ *which at* $q = 0$ *coincides with the usual cup-product, and* $(a_1|...|a_m) = (a_1 * ... * a_m, 1)$. *Beside this, the differential equations* $\hbar q_i \partial q_i \vec{I} = p_i * \vec{I}, i = 1, ..., k$, *for a vector-function* $I(q) \in H^*(X)$ *form a compatible system (i. e. a flat connection) for each* $\hbar$.

The $*$-product is graded if one assigns usual degrees $\deg a_j = \operatorname{codim} A_j$ to the cocycles and non-trivial degrees $\deg q_i = 2d_i$ to the parameters where $d_1 p_1 + ... + d_k p_k = c_1(X)$.

Actual definitions of Gromov–Witten invariants involve non-integrable perturbations of the complex structure on $\mathbb{C}P^1 \times X$, and rigorous computation of quantum cohomology is a non-trivial problem. The following examples, except for the first one, are rather reasonable conjectures than theorems.

Examples. 1) $H_q^*(\mathbb{C}P^{n-1}) \simeq \mathbb{C}[p, q]/(p^n - q)$, and the differential system is equivalent to the scalar equation $\hbar^n d^n I/dt^n = e^t I$ where $t = \log q$. The intersection pairing is given by the residue formula $\oint a(p)b(p)dp/(p^n - q)$ and similar formulas hold in all other examples below.

2) For the space F_n of complete flags in $\mathbb{C}^n$, denote A_n the $n \times n$ matrix with $u_1, ..., u_n$ on the diagonal, $q_1, ..., q_{n-1}$ right above, -1's right under the diagonal, and zeroes otherwise. Put $\Sigma_i = trA_n^i$. Then (see [11]) $H_q^*(F_n) \simeq \mathbb{C}[u,q]/(\Sigma)$. In fact Σ_i coincide with conservation laws of a Toda lattice on n particles with potentials $q_i = -e^{t_i - t_{i+1}}$ (see [11]). The question *why* the algebra $H_q^*(F_n)$ is isomorphic to the algebra of functions on the singular invariant variety of the Toda lattice, is open.

3)Let $X = \mathbb{C}^N//T^k$ be a toric manifold obtained by the Marsden – Weinstein reduction from the standard Hermitian space by a subtorus in the maximal torus T^N. Denote (m_{ij}) the integral $k \times N$ matrix of the natural projection $Lie^*T^N \to Lie^*T^k$. Then the quantum cohomology algebra of X is given by the generators $(u_1, ..., u_N, p_1, ..., p_k, q_1, ..., q_k)$ and relations $u_j = \sum_i m_{ij}p_i,\ q_i = \prod_j u^{m_{ij}}$ (V.Batyrev, see also [9] where a discrete version of quantum cohomology of toric manifolds had been computed as a by-product of a symplectic fixed point theorem).

4)Let X^3 be a non-singular quintic in $\mathbb{C}P^4$. Its hyperplane section p generates in $H^*(X)$ a subalgebra $H^{even} = \mathbb{C}[p]/(p^4)$ with the intersection form $(p^i, 1) = 0$ for $i \neq 3$ and $(p^3, 1) = 5$. Its quantum deformation is almost the same except for $p * p = K(q)p^2$ or, equivalently, $(p * p, p) = K(q)$ where $K(q) = 5 + \sum_{d=0}^{\infty} n_d d^3 q^d/(1 - q^d)$ (see [2]). Here n_d is the number of degree-d rational curves in X: on a generic (almost)-Kahler 3-fold with $c_1 = 0$ rational curves are discrete and all contribute to the quantum cup-product (which now respects the usual grading i. e. $\deg q = 0$). The corresponding differential system is equivalent to $(I''/K(q))'' = 0$ with $' = \hbar q d/dq$. It is degenerate in the sense that it is independent on $\hbar$ and easy to solve, except for the numbers n_d with $d > 3$ are unknown! [1]

In fact we have described only few of all Gromov - Witten invariants (see [18]), which form a complete set of 'correlation functions' of a sigma-model, or A-model of TCFT and when computable, provide algebraic geometry with very non-trivial new enumerative information [13].

3. The mirror conjecture. Mirror Conjecture predicts equivalence of A and B models of $TCFT$ on an algebraic Calabi–Yau manifold to B and A models on its geometrical mirror. In our 'down-to-earth' language: for geometrical mirrors X and Y the differential system of $H_q^{even}(X)$ should coincide with the Picard–Fuchs equation for Y taken in flat coordinates (and vice versa). Authors [6] of this formulation exploited it in order to predict numbers n_d for quintics.

They start with one of the periods $I_1 = \int \omega_\lambda^{3,0} = \sum (5k)!\lambda^k/(k!)^5$, reconstruct the Picard–Fuchs equation: $D^4 I = 5\lambda(5D+1)(5D+2)(5D+3)(5D+4)I$, where $D = \lambda d/d\lambda$, bring it in a neighborhood of the singular point $\lambda = 0$ to the form $(J''/k(q))'' = 0$, conjecture that $k(q) = K(q)$ and find from this $n_1 = 2875, n_2 = 609250, n_3 = 317206375$ (eventually in coherence with available data) and predict n_4

[1] See however [12]

to be 242467530000.

Bringing equation to the simple form involves: 1) finding the solution $I_2 = \log(\lambda)I_1 + \tilde{I}$ with $\tilde{I}$ holomorphic and vanishing at $\lambda = 0$, 2) introducing the new local coordinate $q = \lambda \exp(\tilde{I}/I_1)$ and 3) computing the equation satisfied by $J_i = I_i/I_1, i = 1, 2, 3, 4$, as functions of q.

Our previous discussion suggests the following generalization of the mirror problem:

Is there a natural map (functor?) from (a class of) TCFT-models (symplectic sigma-models, or Frobenius manifolds, or quantum cohomology algebras) to generalized Landau - Ginzburg data? Simpler, *how to solve the differential equations $\hbar dI = A \wedge I$ by means of exponential holomorphic integrals?* We will partially answer this question for the class of toric manifolds.

From such a point of view the Picard–Fuchs equation for Y should have an *intrinsic* interpretation in terms of the problem of computing Gromov-Witten invariants for X. We will point out some.

The (open) question why the above computational procedure (for an equation that had already been intrinsicly attributed to X) yields quantum cohomology of X, is probably related to the problem *in what sense* the above 'functor' is an *involution* on its invariant subset that the class of algebraic Calabi-Yau manifolds seems to constitute.

And all these problems lead to the same question: What is the intrinsic meaning of *solutions* of the quantum cohomology differential system?

4. A project: Equivariant Floer cohomology. Let LX be the space of contractible loops in a compact Kahler manifold X. It inherits the Kahler structures from X and additionally carries the action of S^1 by isometries (translations in the source). The Hamiltonian of this action is the action functional: to a contractible loop it assigns the symplectic area of a disk, contracting the loop, and can be multiple-valued. Doing Morse - Novikov theory for the action functionals $H : \tilde{LX} \to \mathbb{R}$ on the universal covering of LX, one comes to definition of Floer homology FH of X (isomorphic to $H^*(X, \mathbb{C}[q^{\pm 1}])$) [8]: gradient trajectories of H in LX are holomorphic cylinders in X. If one introduces multiplication in FH using the 'map' $LX \times LX \to LX$ of composition of loops (= holomorphic pants in X), it leads to the construction of quantum multiplication in $H_q^*(X)$ (see [11]).

Project: Construct S^1-equivariant Floer cohomology $FH_{S^1}(\tilde{LX})$. Let $\omega_1, ..., \omega_k$ be Kahler forms LX corresponding to the integral basis of Kahler forms on X, let $H_1, ..., H_k$ be corresponding Hamiltonians (of the same S^1-action!) on the covering $\tilde{LX} \to LX$ with the group of covering transformations $\mathbb{Z}^k = H_2(X, \mathbb{Z})$. The generators $q_1, ..., q_k$ of the group $\mathbb{Z}^k$ commute with the S^1-action, preserve the forms ω_i lifted to $\tilde{LX}$ from LX, but transform H_i : $q_i^*(H_j) = H_j - \delta_{ij}$.

Denote $\mathbb{C}[\hbar]$ the coefficient algebra $H^*(BS^1) = H^*(\mathbb{C}P^\infty)$ of the equivariant theory and introduce *Duistermaat - Heckman* equivariantly closed forms $p_i = \omega_i + \hbar H_i$ see [3]).

Proposition: *$p_i q_j - q_j p_i = \hbar q_i \delta_{ij}$ as operators on an equivariant Floer complex.*

Corollary: *$FH_{S^1}(\tilde{LX})$ should carry the module structure over the algebra $\mathcal{D}$ of differential operators generated by $q_i = e^{t_i}\cdot$, and $p_j = \hbar \partial/\partial t_j$.*

A semi-classical limit $\hbar \to 0$ should give rise to the subalgebra in $H_q^*(X)$ generated by the Kahler classes p_i and q_i. In particular relations between them should describe a lagrangian variety with respect to the Poisson bracket $\{p_i, q_j\} = q_i \delta_{ij}$ — the characteristic variety of the $\mathcal{D}$-module.

5. Realization: Toric manifolds. Holomorphic maps $\mathbb{C}P^1 \to X$ to a toric manifold $X = \mathbb{C}^N//T^k$ can be described as equivariant maps $\mathbb{C}^2 \to \mathbb{C}^N$. This compactifies the map spaces up to toric manifolds $X_d = \mathbb{C}^{N+D}//T^k$. Here the homology class d of maps can be identified with an integral point in $LieT^k$ such that $\forall j = 1, ..., N,\ D_j = \sum_i m_{ij} d_i \geq 0$, and $D = \sum D_j$. We can interpret the map spaces as approximations of LX by algebraic loops ($S^1_{\mathbb{C}} \subset \mathbb{C}P^1$), define $FH_{S^1}(X)$ as a certain limit (see [10]) of $H^*_{S^1}(X_d)$ and using the explicit toric description of X_d, compute the $\mathcal{D}$-module.

The algebra $H = H^*(X)$ is generated by the integral Kahler classes $P_1, ..., P_k$ (See [3]). Denote $(\cdot,\cdot)$ intersection pairing on H, $\Omega : H \to H$ — the automorphism generated by $P_i \mapsto -P_i$. Introduce notations: $U_j = \Sigma m_{ij} P_i$, $D_j = \Sigma m_{ij} d_i$, $t_i = \log q_i$, $\partial_j = \hbar \Sigma m_{ij} \partial/\partial t_i$, $\Delta_j^r = \partial_j(\partial_j - \hbar)...(\partial_j - (r-1)\hbar)$ and for any $M \in \mathbb{Z}$ put $M[x]! = \hbar^M \prod_{m=-\infty}^{M}(x+m)/\prod_{m=-\infty}^{0}(x+m)$.

Theorem 1.[10] *Suppose $c_1(X) > 0$. Then*

1. $FH_{S^1} \simeq \mathcal{D}/\mathcal{J}$ where the left $\mathcal{D}$-ideal $\mathcal{J}$ is generated by all operators $\Delta_1^{r_1}...\Delta_N^{r_N} - q^m \Delta_1^{l_1}...\Delta_N^{l_N}$ with $r_j \geq 0$, $l_j \geq 0$ and $r_j - l_j = D_j$.

2. The kernel of this linear differential system is generated by the components of the following vector-function of t with values in the cohomology algebra H: $\vec{f_\hbar}(t) = \hbar^{k-N} e^{Pt} \Sigma_{d \in \mathbb{Z}^k} e^{dt}/D_1[U_1]!...D_N[U_N]!$.

3. $\sum_d e^{d\tau} \int_{X_d} e^{p(t-\tau)/\hbar} = \hbar^{N-k}(\vec{f_\hbar}(t), \Omega \vec{f}_{-\hbar}(\tau))$,
where p_i are Duistermaat - Heckman forms $\omega_i + \hbar H_i$ on each X_d corresponding to our basis in $Lie^*T^k \simeq H^2(X_d)$ and the S^1-action.

4. Suitable limits to $\hbar = 0$ give rise to the algebra $H_q^*(X)$ and a generating series for symplectic volumes of X_d.

Example. For $k = 1, N = n$ and $(m_{ij}) = (1, ..., 1)$ we get $X = \mathbb{C}P^{n-1}$. Then $P^n = 0$ and components of $\vec{f} = e^{Pt} \sum_{d=0}^{\infty} e^{dt}/[(P+1)...(P+d)]^n \hbar^{nd}$ give all n solutions of $(\hbar d/dt)^n I = e^t I$, and the first one ($P = 0$) is $\sum q^d/(d!)^n \hbar^{nd}$.

6. Toric complete intersections. Given T^k-invariant homogeneous polynomials on $\mathbb{C}^N$, one can plug components of a rational curve $\mathbb{C}^2 \to \mathbb{C}^N$ into them and equate to zero identically. In the spaces X_d the solutions form the zero locus of a $PSL_2(\mathbb{C})$-invariant holomorphic section of a suitable equivariant vector bundle. If such sections were transverse to the zero section the loci would represent equivariant Euler classes of these bundles. One may hope to reconstruct the $\mathcal{D}$-modules and quantum cohomology algebras of complete intersections $X' \subset X$ from such classes, substituting them for fundamental cycles of the map spaces X'_d.

For simplicity let us consider the case of Calabi–Yau complete intersections in $X = \mathbb{C}P^{n-1}$. Let $l_1, ..., l_r > 0$ be Chern numbers of r line bundles with $l_1 + ... + l_r = n$. Introduce the algebra $H = \mathbb{C}[P]/(P^{n-r})$ with the intersection pairing $(P^{n-r-1}, 1) = l_1...l_r$ (the image of $H^*(X) \to H^*(X')$) and denote $E^l_d(p, \hbar)$ the S^1-equivariant Euler class of that 'suitable' vector bundle over X_d.

Theorem 2. [10]
$\Sigma_{d=0}^{\infty} e^{d\tau} \int_{X_d} e^{p(t-\tau)/\hbar} E^l_d(p,\hbar) = (-1)^{n-1} \hbar^{1+r-n} (\vec{g}_l(t), \Omega \vec{g}_l(\tau))$ *where*
$\vec{g}_l = e^{Pt} \sum_{d=0}^{\infty} e^{dt} (l_1 d)[P]!...(l_r d)[P]!/(d[P]!)^n$. *The $n-r$ components of $\vec{g}_l$ in H provide a complete solution to the differential equation ($D = d/dt$):*
$D^{n-r} I = l_1...l_r e^t (l_1 D + 1)...(l_1 D + l_1 - 1)...(l_r D + 1)...(l_r D + l_r - 1) I.$

This is exactly the equation that was found in [5] as the *Picard–Fuchs* equation for mirrors of projective Calabi–Yau complete intersections, satisfied by the 'hypergeometric' series $\sum_d l_1!...l_r! q^d/(d!)^n$.

In particular we have obtained the equation $D^4 I = 5e^t(5D+1)...(5D+4)I$ entirely in topological terms of map spaces and not as a Picard – Fuchs equation of a Landau – Ginzburg model. Furthermore, its solution
$e^P t \sum q^d (5P+1)...(5P+5d)/(P+1)^5...(P+d)^5$, rewritten as
$e^{Pt}(G_1 + G_2 P + G_3 P^2 + G_4 P^3) = G_1(q) + P(G_1(q) \log q + G_2(q)) + ...$,
yields I_1 as G_1 and $\tilde{I}$ as G_2.

Thus each coefficient of $G_1, ..., G_4$ should hide some enumerative information about rational curves in $\mathbb{C}P^4$ relative to (one or many) quintics: what appeared meaningless in the Picard-Fuchs equation due to accidental choice of the coordinate λ in the family of quintic-mirrors, turns out related directly to the *exterior* geometry of quintics in $\mathbb{C}P^4$.

Problem: Recover this information.

7. Integral representations. It is not surprizing that toric geometry provides integral formulas for some hypergeometric series. However they will illuminate possible nature of mirror manifolds.

Definition. A function $F : E \to \mathbb{C}$ on the fibered space $\pi : E \to B$ *generates* the lagrangian variety $L = \{(p,t) \in T^*B \| \exists x \in \pi^{-1}(t) : dF|_x = \pi^*(p)\}$.

Lemma. Theorem 1 with $k = N$ and $m_{ij} = \delta_{ij}$ formally gives $P = 0$ and $\vec{f} = \sum_{d \in \mathbb{Z}_+^N} e^{dt}/d_1!...d_N! \hbar^{N|d|} = \exp(\sum e^{t_j}/\hbar)$.

Theorem 3 [10]. *Let $X = \mathbb{C}^N//T^k$ be a compact toric manifold with $c_1 > 0$.*

Then

1. The quantum cohomology algebra $H_q^*(X)$ is the algebra of functions on the lagrangian variety generated by $F = u_1 + ... + u_N : \mathbb{C}^N \to \mathbb{C}$ with $\pi : u \mapsto q$ given by $q_i = \prod u_j^{m_{ij}}$.

2. Take the holomorphic volume form ω_q on $\pi^{-1}(q)$ equal $(\wedge_j du_j/u_j)/(\wedge_i dq_i/q_i)$. Then integrals

$$I(\log q) = \int_{\Gamma \subset \pi^{-1}(q)} \omega_q \; e^{(u_1+...+u_N)/\hbar}$$

over cycles Γ corresponding to $\dim H_q^*(X)$ critical points of $F|\pi^{-1}(q)$ provide a complete set of solutions to the differential system of Theorem 1.1.

Notice that $\dim \pi^{-1}(q) = N - k = \dim_{\mathbb{C}} X$. According to our formulation of the mirror problem we should call the Landau–Ginzburg data $(E \to B, F, \omega)$ *a family mirror-symmetric to the toric manifold X.*

Furthermore, put $X'_q = F^{-1}(1) \cap \pi^{-1}(q)$, $\omega'_q = \omega_q / d(F|\pi^{-1}(q))$.

Theorem 4 [5, 10] *All solutions of the differential equation of Theorem 2 with $r = 1$ are integrals $\int \omega'_q$ over compact cycles $\Gamma' \in X'_q$.*

In order to obtain the same result for $r > 1$ one should split $u_1 + ... + u_N$ into r sums of length $l_1, ..., l_r$ and consider the sums as equations of a complete intersection X' in the fibres of π.

Theorem 4 matches well to remarkable Batyrev's construction [4] of geometrical mirror pairs of toric hypersurfaces: fibers $\pi^{-1}(q)$ are the complex tori which, when suitably compactified into a toric variety, meet $F^{-1}(1)$ along Batyrev's Calabi–Yau hypersurface, and ω' extends to its holomorphic volume form. Thus Batyrev's mirrors of toric hypersurfaces are hypersurfaces in the mirrors of their ambient toric manifolds.

Example. Replace the homogeneous equation $q^{1/5}(x_1^5 + ... + x_5^5) = x_1 ... x_5$ with the affine equation in $\mathbb{Z}_5^4$-invariant coordinates $u_i = q^{1/5} x_i^5 / x_1 ... x_5$. Then $u_1 ... u_5 = q$, and the equation is $u_1 + ... + u_5 = 1$. This corresponds to our matrix $(m_{ij}) = (1, 1, 1, 1, 1)$.

8. Homological geometry. Along with the differential equations of Theorems 1,2 the above integral formulas have intrinsic cohomological meaning in toric geometry. In fact the description of $H_q^*(X)$ by $N + k$ generators and relations is a q-deformation of the following similar description of $H^*(X)$. A symplectic quotient $\mathbb{C}^N//T^k$ can be identified with the free quotient $\check{\mathbb{C}}^N/T^k_{\mathbb{C}}$ of some domain in $\mathbb{C}^N$, and $H^*(\mathbb{C}^N//T^k)$ — with the equivariant cohomology $H^*_{T^k}(\check{\mathbb{C}}^N)$. One begins with $H^*_{T^N}(\mathbb{C}^N) = H^*(BT^N) = \mathbb{C}[u]$, then computes $H^*_{T^N}(\check{\mathbb{C}}^N)$ which causes factorisation by some 'multiplicative' ideal, and finally reduces the group T^N to T^k which imposes the additive relations.

Therefore our $(u_1, ..., u_N)$ are in fact characteristic classes of T^N, and the function $F = \sum u_j$ is the *universal* 1-st Chern class c_1 (of all toric manifolds – quotients of $\mathbb{C}^N$).

Our solutions to the quantum cohomology differential systems are given by integrals in $SpecH^*(BT^N)$ over Morse theoretic cycles of the function $\mathrm{Re}(c_1)$. It suggests that in general mirror manifolds should live in some cohomologies of each other.

The integral formulas can be recovered from our $\mathcal{D}$-module approach: one should first compute 'equivariant Floer cohomology' which are equivariant also with respect to the maximal torus T^N/T^k of $Aut(X)$, and then get rid of this extra-structure. The first step adds variables, the second — expresses $\vec{f}$ as a De Rham class in excessive variables.

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