QUANTUM COHOMOLOGY OF FLAG MANIFOLDS AND TODA LATTICES

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Abstract

We discuss relations of Vafa's quantum cohomology with Floer's homology theory, introduce equivariant quantum cohomology, formulate some conjectures about its general properties and, on the basis of these conjectures, compute quantum cohomology algebras of the flag manifolds. The answer turns out to coincide with the algebra of regular functions on an invariant lagrangian variety of a Toda lattice.

1 Introduction

Quantum cohomology of compact complex Kahler manifolds was introduced by C.Vafa [V] in connection with the theory of mirror manifolds.

By Vafa's definition, the quantum cohomology $QH^*(X)$ of a compact Kahler manifold X is a certain deformation of the cup-product multiplication in the ordinary cohomology of X. Let a, b, c be three cycles in X representing three given cohomology classes by Poincare duality. One defines the *quantum cup-product* $a * b$ by specifying its intersection indices with all c. Namely

$$\langle a * b, c \rangle = \sum_{\text{degree } d \text{ discrete holomorphic maps: } (\mathbb{C}P^1, 0, 1, \infty) \to (X, a, b, c)} \pm q^d.$$

In other words, the intersection index takes in account rational parametrized curves in X with the three marked points — images of 0,1 and ∞ — on the three cycles a, b and c respectively.

This definition needs some explanations.

1. First of all, a rational curve contributes to the intersection index only if it is "discrete" which means, by definition, that

$$c(d) + \dim X = \operatorname{codim} a + \operatorname{codim} b + \operatorname{codim} c$$

where $c(d)$ is the first Chern class c of (the tangent bundle to) X evaluated on the homology class d of the curve, $\dim X$ is the complex dimension of X, and *codim* on the RHS stand for

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degrees of the cohomology classes represented by a, b, c, also counted in *complex* units (so that a real hypersurface has codimension $1/2$). The meaning of the LHS is the dimension of the parameter space of such curves predicted by the classical Riemann–Roch formula, while the RHS is the number of constraints imposed at $0, 1$ and ∞. Thus in the situation of "general position", when the Riemann–Roch prediction is correct (and under some further transversality assumptions) the "discrete" curves can really be treated as isolated intersections and contribute to $\langle a * b, c \rangle$ by $\pm q^d$ each.

2. Here "q^d" is, formally speaking, the homology class of the rational curve and therefore the intersection index as a whole is an element of a group ring of the lattice $H_2(X, \mathbb{Z}) \cap H_{1,1}(X, \mathbb{C})$. The notation q^d is chosen simply to "tame" the group ring by means of coordinates on the lattice. If we choose a basis of Kähler forms $\omega_1, ..., \omega_k$ in $H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{C})$ and express the homology class of a rational curve S by the string $d = (d_1, ..., d_k)$ of its coordinates in the dual basis (so that $d_i = \int_S \omega_i \geq 0$) then the element q^d of the group ring can be identified with the monomial $q_1^{d_1}...q_k^{d_k}$ of the formal variables $(q_1, ..., q_k)$, and the intersection index $\langle a * b, c \rangle$ becomes a formal series in q.

3. The constant term of this series counts *constant* rational curves with the marked points in the cycles a, b, c, i. e. it counts ordinary intersection points. The signs $\pm$ should be chosen in such a way that this term is the ordinary triple intersection index $\langle a \cap b, c \rangle$ of the cycles.

4. About the higher degree terms (they are called "instanton corrections" to the classical intersection index) we only tell here that their signs $\pm$ are defined to be pluses only in the case when the cycles a, b, c are complex submanifolds in X (while the general case will be briefly discussed in 2.3). In any way, the instanton corrections provide a q-deformation of the classical triple intersection index.

5. The double intersection index $\langle a, c \rangle$ of any two cycles, by definition, coincides with the ordinary non-degenerate Poincare pairing, and one can recover the quantum cup-product $a * b$ from the triple pairings as an element of $H^*(X, \mathbb{Z}[[q]])$.

The above construction of the quantum cohomology ring is lacking of many ingredients which could possibly make it mathematically rigorous, and we will touch some mathematical aspects of the problem in the next section. On the other hand, Vafa's construction is strongly supported by general ideology of Conformal Topological Field Theory and provides mathematicians with a bunch of interrelated conjectures. In particular, according to these conjectures, the quantum cup-product

- can be defined rigorously;

- is associative and skew-commutative;

- is a q-deformation of the classical cup-product;

- respects the usual grading in the cohomology provided that one assigns the following nontrivial degrees to the parameters of the deformation: $\deg q^d = c(d)$ (in complex units).

In this paper, we do not have any intention to justify these properties mathematically. Instead, our objective is to compute the quantum cohomology algebras of the classical flag manifolds *in the assumption that their properties expected on the basis of Topological Field Theory are valid.* Therefore the results obtained in this way, while "physical theorems", have the status of mathematical conjectures, or better to say *conditional* theorems contingent to the general conjectures about quantum cohomology of Kahler manifolds. With this reservation in mind we formulate below the results of our computation as *theorems.*

Let F_{n+1} denote the manifold of complete flags

$$\mathbb{C}^1 \subset ... \subset \mathbb{C}^n$$

in $\mathbb{C}^{n+1}$. The cohomology algebra $H^*(F_{n+1})$ is known to be canonically isomorphic to the quotient of the polynomial algebra $\mathbb{Z}[u_0, ..., u_n]$ in $n+1$ indeterminates by the ideal generated by the elementary symmetric polynomials $\sigma_1(u), ..., \sigma_{n+1}(u)$. The generators u_i are in fact the 1-st Chern classes of the tautological line bundles over the flag manifold with the fiber $\mathbb{C}^{i+1}/\mathbb{C}^i$. They are constrained by $u_0 + ... + u_n = 0$ and can be expressed through another basis as $u_i = p_i - p_{i+1}$. The generators $(p_1, ..., p_n)$ are 1-st Chern classes of the *determinant* line bundles with the fiber $\Lambda^*\mathbb{C}^i$ over a point $\mathbb{C}^1 \subset ... \subset \mathbb{C}^n$ of the flag manifold. These determinant line bundles are *non-negative* and the classes p_i span the edges of the (simplicial) Kahler cone in the 2-nd cohomology of F_{n+1}. For a rational curve $S \subset F_{n+1}$ we define its *degree* $d = (d_1, ..., d_n)$ with respect the coordinates p_i as $d_i = \langle p_i, [S] \rangle \geq 0$. Now the homology class of the curve is represented by the monomial $q^d = q_1^{d_1}...q_n^{d_n}$.

In order to describe the quantum cohomology algebra $QH^*(F_{n+1})$ it suffices therefore to exhibit the corresponding deformation of elementary symmetric polynomials of $u_0, ..., u_n$ by the parameters $q_1, ..., q_n$. Notice that while the degrees of u_i are equal 1, the degrees of all q_i are equal 2 (since the 1-st Chern class of the flag manifold is $c = 2(p_1 + ... + p_n)$), and the deformation should be homogeneous with respect to this grading.

Consider the diagonal matrix with $u_0, ..., u_n$ on the diagonal. Then the coefficients of its characteristic polynomial are elementary symmetric functions of u.

Consider another $(n+1) \times (n+1)$ matrix, denoted A_n,

$$A_n = \begin{bmatrix} u_0 & q_1 & 0 & ... & 0 \\ -1 & u_1 & q_2 & ... & 0 \\ 0 & -1 & u_3 & ... & 0 \\ & . & . & . & \\ 0 & ... & 0 & -1 & u_n \end{bmatrix}$$

with u_i on the diagonal, q_i — right above, and -1's — right under the diagonal. Then the coefficients of its characteristic polynomial are the deformations in question of the elementary symmetric functions:

Theorem 1. *The quantum cohomology algebra $QH^*(F_{n+1})$ of the flag manifold is canonically isomorphic to the quotient of the polynomial algebra $\mathbb{Z}[u_0, ..., u_n, q_1, ..., q_n]$ by the ideal generated by coefficients of the characteristic polynomial of the matrix A_n.*

Specialists on complete integrable systems will recognize in this answer something very familiar: in fact the coefficients of $\det(A_n + \lambda)$ are conservation laws of a Toda lattice.

Namely, introduce "configuration" variables $(x_0, ..., x_n)$ of $n + 1$ consecutive unit masses on the line with $q_i = \exp(x_i - x_{i-1})$ in the role of potential energy of neighbors. Then

$$\frac{1}{2}\operatorname{tr}(A_n^2) = \frac{1}{2}\sum u_i^2 - \sum e^{x_i - x_{i-1}}$$

is the Hamiltonian of the classical Toda lattice (with incorrect sign of the potential however), and $\operatorname{tr}(A_n^i)$, $i = 1, ..., n + 1$, is the complete set of commuting first integrals.

Corollary. *The quantum cohomology algebra of the flag manifold F_{n+1} is isomorphic to the algebra of functions on the common zero level of the first integrals of the classical Toda lattice.*

Making comments on the theme "How much surprising is the result?" we should say that one might not expect quantum cohomology of flag manifolds to have no connections with other known objects attributed to flag manifolds. Moreover, Topological Field Theory predicts deep relations (see for instance [D],[W]) of moduli spaces of rational curves in Kahler manifolds with hierarchies of integrable systems. Moreover, Toda lattices have already occurred [CV] — in a "less surprising" manner — in some dynamical problem related to quantum cohomology of projective spaces. Nevertheless the authors should confess they did not foresee this particular relation when started the computation, and they do not know now how the answer can be predicted. However some partial explanations should be given right away.

First of all, it can be viewed accidental that the relations in quantum cohomology of flag manifolds Poisson-commute. What is not accidental at all is that they Poisson-commute *modulo the relations themselves*. Indeed, according to general theory (see 2.4) quantum cohomology algebra of a Kahler manifold in some sense always is (or at least related to) the algebra of functions on some lagrangian variety in the cotangent bundle of some torus. The parameters q_i of the quantum deformation are multiplicative coordinates on the torus. In the case of F_{n+1} the cotangent bundle provided with the coordinates $q_1, ..., q_n \neq 0, p_1, ..., p_n$ (in above notations) has the canonical symplectic form

$$dp_1 \wedge \frac{dq_1}{q_1} + ... + dp_n \wedge \frac{dq_n}{q_n},$$

and the algebra $QH^*(F_{n+1}, \mathbb{C})$ must be the algebra of regular functions on some quasi - homogeneous lagrangian subvariety L. In view of the group-theoretic nature of Toda lattices [R], our theorem leads to the following geometrical description of L.

Let $G = SL_{n+1}(\mathbb{C})$, N_+ and N_- be its strictly lower- and upper-triangular subgroups. Make N_+ and N_- act respectively by left and right translations on the cotangent bundle T^*G of the group and consider the momentum map $J : T^*G \to Lie^*(N_+ \times N_-)$ of the action. The trace inner product $\operatorname{tr} AB$ on the matrix algebra identifies the dual of the Lie algebra of $N_+ \times N_-$ with the quotient of the space of all square $(n+1)$-matrices by the subspace of all diagonal matrices. Pick

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the value of the momentum map as specified by the matrix

$$P = \begin{bmatrix} * & 1 & 0 & 0 & ... \\ 1 & * & 1 & 0 & ... \\ 0 & 1 & * & 1 & ... \\ & . & . & . & \\ ... & 0 & 0 & 1 & * \end{bmatrix}$$

(0's everywhere except 1's right above and under the diagonal) and make the symplectic reduction on this level of the momentum map. The reduced phase space

$$M_P = T^*G//_P(N_+ \times N_-) = J^{-1}(P)/(N_+ \times N_-)$$

can be naturally identified with the cotangent bundle of the maximal torus in G. Now, consider the cone $C \subset Lie\,G$ of all nilpotent traceless matrices. The product

$$C \times G \subset (Lie\,G) \times G \;=\; T^*G$$

is a bi-invariant involutive subvariety. Its symplectic reduction

$$L = [J^{-1}(P) \cap (C \times G)]/(N_+ \times N_-) \subset M_P$$

is in fact a lagrangian subvariety in the reduced phase space.

Corollary. *The quantum cohomology algebra $QH^*(F_{n+1}, \mathbb{C})$ is isomorphic to the algebra of regular functions on the lagrangian variety L.*

We should augment this corollary with an open question: *Why* the quantum cohomology algebra of the flag manifold G/B_- is isomorphic to the algebra of regular functions on the lagrangian variety L? We would expect that a natural answer to this question will come along with a better understanding of the general mirror symmetry phenomena (cf. [G3]).

The second argument that partially explains the theorem comes from its proof. Our computation of quantum cohomology of flag manifolds is based in fact on induction on n. It turns out however that the induction assumption that quantum cohomology of F_{m+1} with $m < n$ is known, is insufficient for our purpose. What we really need is an *equivariant* version of quantum cohomology of flag manifolds considered as homogeneous spaces of unitary groups. Similarly to ordinary equivariant cohomology of a U-space X, quantum equivariant cohomology can be defined (with similar reservations) as a skew-commutative associative algebra over the ring of characteristic classes of the compact Lie group U.

In the case of $U = U_{n+1}$ (acting on the flag manifold F_{n+1}), we deal with the algebra $\mathbb{Z}[c_1, ..., c_{n+1}]$ of usual Chern classes, and the ordinary equivariant cohomology of the flag manifold is known to coincide with the polynomial algebra $\mathbb{Z}[u_0, ..., u_n]$ of characteristic classes of the maximal torus $T^{n+1} \subset U_{n+1}$ considered however as a module over the subalgebra of Chern classes

$$c_i = \sigma_i(u_0, ..., u_n), \; i = 1, ..., n+1$$

— elementary symmetric functions of u.

In the same manner as $H^*(F_{n+1})$ is obtained from the equivariant cohomology $H^*_{U_{n+1}}(F_{n+1})$ by specialization $c_1 = ... = c_{n+1} = 0$, we deduce our theorem on quantum cohomology of flag manifolds from a more general result describing their equivariant quantum cohomology.

Theorem 2. *The equivariant quantum cohomology algebra $QH^*_{U_{n+1}}(F_{n+1})$ is canonically isomorphic to the quotient of the polynomial algebra*

$$\mathbb{Z}[u_0, ..., u_n, q_1, ..., q_n, c_1, ..., c_{n+1}]$$

by the ideal of relations obtained by equating the coefficients of the following polynomials in λ:

$$\det(A_n + \lambda) = \lambda^{n+1} + c_1\lambda^n + ... + c_n\lambda + c_{n+1}.$$

In other words, it is the free polynomial algebra in u and q but the subalgebra of Chern classes, instead of symmetric functions of u, consists of their "quantum deformations" from the previous theorem — first integrals of the Toda lattice.

Now we can figure out, why one might *a priori* expect quantum cohomology of flag manifolds to be related with at least *some* integrable system.

According to our general theory (see 3.8), equivariant quantum cohomology of a compact Kahler U-manifold X is an algebra of functions on a lagrangian subvariety $\mathcal{L}$ in a Poisson manifold with U-characteristic classes in the role of Casimir functions. Poisson structure lives in the space with coordinates $(q_1, ..., q_n, p_1, ..., p_n, c_1, ..., c_{n+1})$ and is given by the formula

$$q_1\frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} + ... + q_n\frac{\partial}{\partial p_n} \wedge \frac{\partial}{\partial q_n}$$

so that the symplectic leaves $\vec{c} = const$ are in fact all isomorphic to the cotangent bunle of the q-torus described above.

Our point now is that although equating Chern classes to non-zero constants makes little "cohomological" sense, the ideal of $\mathcal{L}$ is *a priori* a Poisson ideal, and therefore intersections of $\mathcal{L}$ with the symplectic leaves can be interpreted as a $\vec{c}$-parametric family of lagrangian submanifolds in *the same* symplectic space — the cotangent bundle of the torus.

Moreover, since the ideal of relations is generated by quasi-homogeneous q-deformations of the classical relations $c_i = \sigma_i(u)$, equations of the lagrangian submanifolds have the following *triangular* form

$$c_i = C_i(u, q, c_1, ..., c_{i-1}), \ i = 1, ..., n+1$$

and can be resolved with respect to c_i as $c_i = c_i(u, q)$.

This means that the lagrangian submanifolds fit nicely into the phase space as leaves of a lagrangian foliation — common levels of the functions $c_i(u, q), i = 1, ..., n+1$, which are therefore in involution, — and the lagrangian variety L is a singular zero leaf of this foliation.

Our description of quantum (equivariant) cohomology of flag manifolds would be incomplete without a formula for the intersection pairing (see 3.4)

$$\langle \cdot, \cdot \rangle : \ QH^*_{U_{n+1}}(F_{n+1}, \mathbb{C}) \otimes_{\mathbb{C}[c]} QH^*_{U_{n+1}}(F_{n+1}, \mathbb{C}) \to \mathbb{C}[c].$$

Denote $\Sigma_i(u_0, ..., u_n, q_1, ..., q_n)$, $i = 1, ..., n+1$, the quantum deformation of elementary symmetric functions $\sigma_i(u)$ from Theorem 1 (i. e. the first integrals of the Toda lattice). Let $\varphi, \psi \in \mathbb{C}[u, q, c]$ be two polynomials considered as representatives of cohomology classes from $H^*_{U_{n+1}}(F_{n+1})$.

Theorem 3.

$$\langle [\varphi], [\psi] \rangle(c, q) = \frac{1}{(2\pi i)^{n+1}} \int \frac{\varphi(u, q, c)\psi(u, q, c)du_0 \wedge ... \wedge du_n}{(\Sigma_1(u, q) - c_1)...(\Sigma_{n+1}(u, q) - c_{n+1})}.$$

The integral here can be replaced by the total sum of $(n + 1)!$ residues in the u-space. In order to obtain the intersection pairing in non-equivariant cohomology $QH^*(F_n)$ it suffices to put $c_1 = ... = c_{n+1} = 0$ in this formula.

Consider the basis $p_1, ..., p_n$ of non-negative $(1, 1)$-classes on F_{n+1}, $u_i = p_i - p_{i+1}$. Then $(z, p) = z_1 p_1 + ... + z_n p_n$ with $z_i > 0$ is represented by a Kahler form, and $\exp(z, p)$ can be considered as a non-homogeneous differential form whose degree (k, k) term measures k-dimensional Kahler volume. The corresponding *quantum generating volume function* (see 2.3):

$$V(z, q) = \frac{1}{(2\pi i)^n} \int \frac{\exp(z, p)\, dp_1 \wedge ... \wedge dp_n}{\Pi_{j=1}^n (\Sigma_{j+1}(u(p), q))}$$

has the geometrical meaning of the total Kahler volume of the 'q-weighted' space

$$\mathcal{M} = \cup_d q^d \mathcal{M}_d$$

of holomorphic maps $\mathbb{C}P^1 \to F_{n+1}$ of all degrees d. The volume is computed in fact with respect to the Kahler form induced by (z, p) on the *loop space* LF_{n+1} where $\mathcal{M}$ can be naturally embedded. Combining our conjectures about general properties of quantum cohomology with the 'conditional' Theorem 3 we come to the following 'unconditional' prediction.

Conjecture. *Kahler volume of the space of parametrized rational curves of degree $d = (d_1, ..., d_n)$ with respect to the Kahler form with periods $z_1, ..., z_n$ on the flag manifold F_{n+1} equals*

$$\mathrm{Vol}_z(\mathcal{M}_d) = \frac{1}{d_1!...d_n!}\left(\frac{\partial}{\partial q_1}\right)^{d_1}...\left(\frac{\partial}{\partial q_n}\right)^{d_n}|_{q=0} V(z, q).$$

At $d = 0$ this formula reduces to the total volume of the flag manifold itself and coincides with the fundamental anti-invariant of the permutation group. The equivariant analogue $V_G(z, q, c)$ of the generating volume function at $q = 0$, $c = \sigma(x_0, ..., x_n)$ turns into the asymptotic character of irreducible representations of $G = U_{n+1}$ with 'large highest weights' proportional to z (it can be found using Duistermaat – Heckmann formula [AB]). It would be interesting to figure out the meaning of such generating volume functions with non-zero q and the role of Toda lattices in representation theory of loop groups. The last question seems to be closely related to the recent paper [FF] on Toda Field Theory.

Structure of this paper. In Section 2 we give a more detailed review of quantum cohomology theory. Although one can find a number of approaches to the general theory in the available

literature (see for instance [W] or a recent preprint [S] where in particular the quantum cohomology of F_3 has been computed), we hope that our point of view is up to certain extent complementary to them. It also should help to clarify our construction of equivariant quantum cohomology (Section 3) as well as those conjectures about its general properties which we exploit in our inductive proof (Section 4) of the theorems formulated in this Introduction.

Conventions. Throughout this paper, we will assume for convenience that all dimensions are counted in complex units, and — for the sake of simplicity — that all considered compact Kahler manifolds are simply-connected.

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2 Quantum cohomology and Floer homology

The objective of this section is to interpret Vafa's construction of quantum cohomology of a compact Kahler manifold as Floer homology of its loop space (to be more precise — of the universal covering of the loop space) provided with multiplication induced by composition of loops.

2.1 Additive structure

Let X be a compact manifold provided with a complex structure J and a riemannian metric $(\cdot, \cdot)$ compatible with the complex structure in the sense that the differential form $\omega = (J\cdot, \cdot)$ is symplectic.

The space LX of contractible (say, smooth) loops $S^1 \to X$ inherits from X the same structures:

- the complex structure $\mathcal{J}$ which transforms a tangent vector (= a vector field $t \mapsto v(t)$ along the loop $t \mapsto \gamma(t)$) to $t \mapsto J(\gamma(t))v(t)$;

- the $\mathcal{J}$-compatible riemannian and symplectic forms

$$(v, w) = \oint (v(t), w(t)) \, dt \ , \ \Omega(v, w) = \oint \omega(v(t), w(t)) \, dt;$$

and additionally carries

- the action of the reparametrization group $Diff(S^1)$ and in particular the circle action generated by the vector field $V : \gamma \mapsto \dot{\gamma}$ on LX; and

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- the *action functional* $\mathcal{A} : \tilde{LX} \to \mathbb{R}$:

$$\mathcal{A}(\gamma) = \int_D \varphi^* \omega$$

which assigns to a loop γ the symplectic area of a disk ($\varphi : D \to X : \varphi|_{S^1 = \partial D} = \gamma$) contracting the loop, and thus is well defined only on the universal covering of LX.

There is a remarkable relation between these structures, namely

1. the circle action is hamiltonian with respect to the symplectic form Ω and the hamilton function is $\mathcal{A}$;

2. the gradient vector field of the action functional relative to the riemannian metric equals $\mathcal{J}V$ and thus the gradient "flow" consists in analytic continuation of loops from the unit real circle $S^1 \subset \mathbb{C} - 0$ to its neighborhood in the complex circle.

By definition, Floer homology $FH_*(X)$ is Morse-theoretic homology of the loop space LX constructed by means of the "Morse function" $\mathcal{A}$ in the spirit of Witten's approach [W2] to the Morse theory, i. e. using *bounded* gradient trajectories joining critical points.

historically Floer homology has been introduced [F1] in order to prove Arnold's symplectic fixed point conjecture and deals with Morse theory of action functionals perturbed by a hamiltonian term. However the homology itself is simpler to compute for the unperturbed action functional $\mathcal{A}$.

In fact the functional $\mathcal{A}$ is a *perfect Morse–Bott–Novikov* function on LX.

Here

- "Novikov" means that it is multiple-valued and thus the Morse–Smale complex should be constructed from the critical points on a covering $\tilde{LX}$ and treated as a module over the group of covering transformations.

- The critical points are in fact constant loops and thus the critical locus of $\mathcal{A}$ on the covering consists of copies of the manifold X itself duplicated as many times as many elements are in the covering transformation group. The critical components are transversally non-degenerate so that $\mathcal{A}$ is a Morse–Bott function.

- The group of covering transformations is in fact the lattice $\mathbb{Z}^k = \pi_2(X) \cap H_2(X, \mathbb{R})$ of spherical periods of closed 2-forms on X and thus the Morse–Smale–Bott–Novikov complex can be identified with the homology group $H_*(X, \mathbb{Z}[q, q^{-1}])$ of X where the coefficient ring is a group ring of the lattice (in the first approximation it can be taken as the ring of Laurent polynomials in k generators $q = (q_1, ..., q_k)$).

- Finally, "perfect" means that the boundary operator in the complex is *zero* so that $FH_*(X) \cong H_*(X, \mathbb{Z}[q^{\pm 1}])$ as a $\mathbb{Z}[q^{\pm 1}]$-module.

The latter statement is due to the fact that $\mathcal{A}$ is the Hamiltonian of a circle action. The Atiyah convexity theorem [A] says in particular that the Hamiltonian of a torus action on a compact symplectic manifold is a perfect Morse–Bott function. A "scientific" explanation [G]

is that the same manifold is the critical set of a function (which leads to the Morse inequality) and the fixed set of a sircle action (which leads to the opposite Smith inequality in equivariant cohomology, see also [G1] where locally hamiltonian torus actions are considered). A geometrical argument behind this property works pretty well in the infinite-dimensional Morse theory if one deals with only bounded trajectories of the gradient flow.

Now we can describe geometrically the Morse–Bott cycles of Floer homology theory. They are enumerated by ordinary cycles in the components of the critical locus. Pick such a component X and a cycle $a \subset X$. The corresponding Morse–Bott (co)cycle $A \subset LX$ is the union of all the gradient trajectories outgoing (resp. ingoing) the critical set a when time $\to -\infty$ ($+\infty$ respectively). Since the gradient flow of $\mathcal{A}$ consists in analytic continuation, we come to the following description of the cycle A:

$A = \{$ boundary values of holomorphic maps of the unit disk $D \subset \mathbb{C}$ to X with the center in $a \subset X$ $\}$.

2.2 Multiplication

After such an informal description of the additive structure in Floer homology it is time to discuss multiplication. There are at least two reasons why analogue of usual cup-product may not exist in Floer's theory:

1. intersections in general position of Morse–Bott cycles in LX which have "semi-infinite" dimension would give rise to the cycles of finite dimension rather than to "semi-infinite" cycles again;

2. finite-dimensional Novikov's cohomology is cohomology with local coefficients determined by periods $\log q$ of the closed 1-form; cup-product of such cohomology is accompanied by tensor multiplication of the local coefficient systems and would give rise to q^2 in the product, instead of q again.

In fact the multiplicative structure in Floer homology is analogous to the convolution in the homology of a Lie group induced by multiplication in the group. The "group" operation on LX consists in composing parametrized loops at the marked point $t = 0$ on the circle S^1. This operation is ill-defined since the loops we consider are *free*. However this operation considered as a *correspondence* can be described by its graph in LX^3, and the convolution multiplication $A * B$ of Morse–Bott cycles can be defined through intersection indices $\langle A * B, C \rangle$ of the products $A \times B \times C \subset LX^3$ with the graph.

By some technical analytical reasons it is more convenient to perturb the graph and consider instead the cycle in LX^3 which consists of triples of loops which are boundary values of a *holomorphic map of "pants"* to X. More generally, one can define multiple products $A_1 * ... * A_N$ through intersection indices $\langle A_1 * ... * A_N, C \rangle$ in LX^{N+1} considering compositions of pants and their holomorphic maps to X.

In more detail, denote Π_N the standard Riemann sphere $\mathbb{C}P^1$ with N disks detached and their boundaries *left* oriented and parametrized by the standard unit circle S^1. Denote Γ_N the cycle in LX^N which consists of N-tuples of boundary values of holomorphic maps $\Pi_N \to X$. For

N given Morse–Bott cycles $A_1, ..., A_N$ in LX define their $\langle A_1|...|A_N \rangle$ as the intersection index of "semi-infinite cycles" $A_1 \times ... \times A_N \subset LX^N$ and Γ_N.

We should make a correction here: the intersection index should be defined as *Novikov's* one. This means that the product $A_1 \times ... \times A_N$ should be considered as a cycle on the diagonal $\mathbb{Z}^k$-covering (LX^N) . An important property of Γ_N is that it has a canonical lifting to this covering: an N-tuple of the boundary values is provided with the homotopy type of the map $\Pi_N \to X$. Novikov's intersection index of two *transversal* cycles A and B on the covering, by definition, assumes values in the group ring of the covering and counts *isolated* intersection points of the cycles projected to the base, with signs and "weights" $q^d \in \mathbb{Z}[q^{\pm 1}]$, where $d \in \mathbb{Z}^k$ is the covering transformation that transforms the preimages in A and B of the intersection point into one another.

Now we can describe geometrically an intersection event of $A_1 \times ... \times A_N$ with Γ_N. The Morse–Novikov cycles A_i correspond to some finite-dimensional cycles a_i in X. An intersection point, on one hand, is an N-tuple of loops which are boundary values of N parametrized holomorphic disks in X with centers respectively in $a_1, ..., a_N$. On the other hand it is the N-tuple of boundary values of a holomorphic map $\Pi_N \to X$. Due to the uniqueness of analytic continuation, the disks and Π_N glue up to a single holomorphic map $\varphi : \mathbb{C}P^1 \to X$ with the centers $x_1, ..., x_N$ of the (formerly detached) disks being mapped to the cycles $a_1, ..., a_N$ respectively. The group element d in the definition of Novikov's intersection index, in our situation measures the difference of homotopy types of the two holomorphic films attached to the N-tuple of loops and equals the homotopy type of the map φ, i. e. the degree of the rational curve $\varphi(\mathbb{C}P^1)$. Thus we come to Vafa's formula:

$$\langle A_1|...|A_N \rangle = \sum_{\substack{\text{isolated holomorphic maps} \\ \varphi : (\mathbb{C}P^1, x_1, ..., x_N) \to (X, a_1, ..., a_N)}} \pm q^{\deg \varphi}.$$

The assumption that the intersected cycles are transversal means that the number of independent holomorphic sections of the induced tangent bundle $\varphi^! T_X$ equals the Euler characteristic $c(d) + \dim X$ prescribed by the Riemann–Roch formula, and the constraints $\varphi(x_i) \in a_i$ are non-degenerate (in the sense of implicit function theorem). Thus the isolatedness implies

$$c(d) + \dim_\mathbb{C} X = \sum_i \text{codim}_\mathbb{C} \, a_i.$$

Notice that holomorphic spheres constrained at two points are never isolated (circle action! By the way it is that geometrical argument that makes $\mathcal{A}$ perfect) and thus the double intersection index $\langle A, B \rangle$ coincides with the non-degenerate Poincare pairing of cycles a, b in X. One can identify a cycle a of codimension α in X with the Poincare-dual cohomology class of degree α. The above formula means that $\langle A_1|...|A_N \rangle$ defines in this way a "quantum" q-valued intersection pairing $H^*(X)^{\otimes N} \to \mathbb{Z}[q^{\pm 1}]$ which respects the usual grading in cohomology provided that $\deg q^d = c(d)$:

$$\deg \langle a_1|...|a_N \rangle = \deg_\mathbb{C} a_1 + ... + \deg_\mathbb{C} a_N - \dim_\mathbb{C} X.$$

The triple "pairing" can be used in order to define the "quantum multiplication" $a * b$:

$$\forall c \quad \langle a * b, c \rangle = \langle a|b|c \rangle.$$

The fact that this multiplication is associative as well as that the multiple pairings can be expressed through $*$-operation and Poincare pairing with the fundamental cycle [**1**] as

$$\langle a_1|...|a_N \rangle = \langle a_1 * ... * a_N, \mathbf{1} \rangle,$$

reduces to the principal axiom of Topological Field Theory: i

> If the surface Π_N is cut by a circle into a union of two surfaces Π_{M+1} and Π_{N-M+1} then the corresponding intersections satisfy
>
> $$\langle a_1|...|a_N \rangle = \sum_j \langle a_1|...|a_M|b_j \rangle \langle c_j|a_{M+1}|...|a_N \rangle$$
>
> where $\sum_j b_j \otimes c_j \in H^*(X \times X)$ is Poincare-dual to the class of the diagonal $X \subset X \times X$).

Rigorous justification of this axiom as well as of correctness of the above definitions is obstructed by a number of highly non-trivial problems.

First of all, in order to bring the cycles in LX^N to transversal position one needs, in general, to perturb the complex structure on X toward almost complex structures, and the whole story begins to depend on Gromov's theory [Gr] of pseudo-holomorphic curves in symplectic manifolds and compactifications of their moduli space.

Even in the additive Floer theory some difficulties (with multiple coverings of holomorphic curves) has not been overcome so far. The situation seems to be simpler, and the difficulty — resolved, in the case of almost Kahler manifolds with positive first Chern class c and almost complex structure close to an integrable one (see [O]). The case of zero first Chern class which also has been worked out [HS], requires Novikov's completion of the group ring $\mathbb{Z}[q^{\pm 1}]$ (Vafa's formula may contain infinite sums).

In the cases when the additive theory can be completed successfully, correctness of the definitions of multiple intersection indices, their skew-commutativity, independence on moduli of surfaces Π_N, on the choice of cycles in the homology classes, and so on, does not seem to exhibit further complications (see [R]).

At the same time, associativity of the quantum multiplication and the axioms of Topological Field Theory have been verified, as far as we know, only in the simplest case of manifolds X with $\pi_2(X) = 0$ (M.Schwartz) where instanton corrections do not occur at all.

2.3 Alternative approaches

We briefly review here some other constructions of quantum cohomology algebras. Later they will be described in more detail in connection with equivariant theory.

First of all, instead of the ill-defined composition map $LX \times LX \to LX$ one can consider a well-defined evaluation map $LX \to LX \times X$:

$$(\text{a loop } t \mapsto \gamma(t)) \mapsto (\gamma \in LX, \gamma(t_0) \in X).$$

It induces a linear map

$$H^*(X) \otimes FH^*(X) \to FH^*(X)$$

and thus makes cohomology classes of X act on the Floer cohomology $H^*(X, \mathbb{Z}[[q^{\pm 1}]])$ of the loop space by $\mathbb{Z}[[q^{\pm 1}]]$-linear operators. These operators, along with operators of multiplication by q, generate some associative skew-commutative operator algebra. Composition of such operators differs in fact from ordinary cup-product in $H^*(X)$. It is not obvious from this point of view even that they should form an algebra closed with respect to composition. However interpretation of matrix elements of such operators in terms of rational curves in X leads directly to Vafa's definition of quantum cup-product. Such a module structure in Floer homology of LX over cohomology of X itself has been exploited many times in the literature on symplectic topology [FW], [Oh], [F2], [H], [G1], [G2] (and in a recent paper [S] on quantum cohomology).

A similar approach, based however on differential forms, was studied in [V]. A closed differential r-form p on X and a density ρ on the unit circle determine a closed differential r-form P on the loop space LX:

$$P|_\gamma(v_1, ..., v_r) = \oint p|_{\gamma(t)}(v_1(t), ..., v_r(t))\rho(t)dt.$$

The ordinary cohomology class of P on LX depends, by the Stokes theorem, only on the class of p on X and on the total "mass" $\oint \rho(t)dt$. However we are going to integrate P over non-compact cycles in LX, so that the Stokes theorem does not apply literally. The cycle we need is denoted $\mathcal{M}_d$ and consists of *algebraic loops of degree d* in X, i. e. degree d holomorphic maps $\mathbb{C}P^1 \to X$ which can be considered as elements of the loop space if we restrict them to the unit circle in $\mathbb{C} - 0$. The cycle $\mathcal{M}_d$ — a "moduli space" of rational curves — can be compactified, after Gromov [G], by reducible curves, and this is a reason to expect that the integral converge. The reducible curves however do not correspond to any loops, and the compactification can not be done inside LX.

One can define quantum intersection pairings as

$$\langle p_1|...|p_N\rangle = \sum_d \pm q^d \oint_{\mathcal{M}_d} P_1 \wedge ... \wedge P_N.$$

assuming the corresponding densities ρ_i, $i = 1, ..., N$, being of unit total mass each and generic. The integrals in this sum can be non-zero only if the total degree $r_1 + ... + r_N$ of the differential form equals the dimension $2(c(d) + \dim X)$ of the cycle $\mathcal{M}_d$ and reduces to $\int_X p_1 \wedge ... \wedge p_N$ for $d = 0$.

The coincidence of such intersection pairings with previously defined ones becomes "obvious" if we interpret them in the spirit of integral geometry. Imagine that the densities ρ_i has been chosen as Dirak δ-functions concentrated at N generic marked points $x_1, ..., x_N$ on the unit circle. Then

$$\int_{\mathcal{M}_d} P_1 \wedge ... \wedge P_N = \int_{\bar{\mathcal{M}}_d} \bar{p}_1 \oplus ... \oplus \bar{p}_N$$

where $\bar{p}_i$ is a differential form on X^N obtained as the pull-back of of p_i on the i-th factor, and $\bar{\mathcal{M}}_d$ is the closure in X^N of the image of the evaluation map

$$(\mathcal{M}_d \subset LX) \to X^N : \ \gamma \mapsto (\gamma(x_1), ..., \gamma(x_N)).$$

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The fundamental class of the complex variety $\bar{\mathcal{M}}_d$ in $H^*(X^d)$ is the same for *generic* marked points. Taking the average value of such integrals, defined by means of δ-densities, over the torus $(S^1)^N$ in the configuration space $(\mathbb{C} - 0)^N$ of marked points we conclude that the quantum intersection pairing of closed forms depends only on their cohomology classes in $H^*(X)$ and does not depend on the densities provided that they are, say, continuous. On the other hand, replacing the forms p_i by their Poincare-dual cycles we find the integral equal to an intersection index in X^N with the "moduli space" $\bar{\mathcal{M}}_d$, and this leads back to the original Vafa's construction — counting rational curves constrained at marked points. Notice that this construction of $\langle p_1|...|p_N\rangle$ as intersection indices in X^N also explains how the signs in Vafa's formula should be chosen.

The last construction of quantum cohomology algebras — via generating volume functions — is most convenient in the case when the ordinary cohomology algebra $H^*(X)$ is generated (as an algebra) by Kahler classes, and will be described below under this assumption. Let $p_1, ..., p_k$ be an integer basis of non-negative $(1,1)$-forms in $H^2(X)$, $p(z) = z_1 p_1 + ... + z_k p_k$ be a general linear combination. If $p(z)$ is a Kahler form on X the corresponding form $P(z)$ is a Kahler form on the loop space LX, and the following formal series

$$V(z,q) = \sum_d q^d \int_{\mathcal{M}_d} \exp(P(z))$$

represents the Kahler volume of the "weighted moduli space"

$$\mathcal{M} = \cup_d \, q^d \mathcal{M}_d,$$

since the terms of the exponential series

$$\exp P = \sum_r \frac{1}{r!} P \wedge ... \wedge P \quad (r \text{ times})$$

represent r-dimensional Kahler volumes with respect to P.

We call $V(z,q)$ *generating volume function* (in fact it is a simplified version of the generating correlation function Φ from CTFT [W],[D],[K]).

It has the following properties:

1. $V(z,q)$ becomes quasi-homogeneous of degree $-\dim X$ if we put $\deg z_i = -1$, $\deg q_i = D_i$ where $c = D_1 p_1 + ... + D_k p_k$ represents the 1-st Chern class of X in the basis $(p_1, ..., p_k)$;

2. $V(z,0) = \int_X \exp(p(z))$ is the volume function of X;

3. quantum intersection indices of the generators $p_1, ..., p_k$ can be expressed in terms of $V(z,q)$ as

$$\langle p_{i_1}|...|p_{i_N}\rangle = \frac{\partial^N}{\partial z_{i_1}...\partial z_{i_N}}|_{z=0} V(z,q)$$

(this is due to the very property of the exponential function).

This last formula implies that one can *define* the quantum cohomology algebra $QH^*(X)$ as the quotient of the polynomial algebra $\mathbb{Z}[p,q]$ by the ideal I of all polynomials $R(p,q)$ such that

$$R(\partial/\partial z_1, ..., \partial/\partial z_k, q_1, ..., q_k)V(z,q) = 0.$$

Example: $QH^*(\mathbb{C}P^1)$. A holomorphic map $\mathbb{C}P^1 \to \mathbb{C}P^1$ of degree d is given by the ratio f/g of two homogeneous polynomials

$$f = \sum a_i x^i y^{d-i}, \ g = \sum b_i x^i y^{d-i}$$

in two variables. This means that the space $\mathcal{M}_d$ of such maps compactifies to the complex projective space $\mathbb{C}P^{2d+1}$. Let p be the Fubini Kahler form on the target $\mathbb{C}P^1$. It is obtained from the form

$$\partial\bar{\partial}\log(f\bar{f} + g\bar{g})$$

in homogeneous coordinates (f, g). The corresponding Kahler form P on $\mathcal{M}_d \subset L(\mathbb{C}P^1)$ is similarly obtained from

$$\partial\bar{\partial}\log(f\bar{f} + g\bar{g})|_{(x,y)=(e^{it},1)}$$

as their mean value over t. At $t = 0$ this gives

$$\partial\bar{\partial}\log[|\sum a_i|^2 + |\sum b_i|^2]$$

and leads to a non-negative $(1,1)$-form which extends to $\mathbb{C}P^{2d+1}$ and represents there a generator of $H^2(\mathbb{C}P^{2d+1}) \cong \mathbb{Z}$. The same properties hold for all t, and thus P represents the class the of Fubini form on $\mathbb{C}P^{2d+1}$. We conclude that

$$V(z, q) = \sum_{d=0}^{\infty} \frac{z^{2d+1}}{(2d+1)!} q^d.$$

It is easy to see that the ideal I of polynomials $F(\partial/\partial z, q)$ annihilating V is generated by $(\partial/\partial z)^2 - q$ and therefore

$$QH^*(\mathbb{C}P^1) = \mathbb{Z}[p, q]/(p^2 - q).$$

We find *a posteriori* that it is indeed a q-deformation of the classical cohomology ring $H^*(\mathbb{C}P^1) = \mathbb{Z}[p]/(p^2)$.

2.4 Characteristic lagrangian variety

Keeping the assumption, that cohomology algebra of X is generated by Kahler classes, and the notations introduced in the end of 2.3, we describe here $QH^*(X, \mathbb{C})$ as the algebra of functions on some lagrangian variety.

Since the quantum cohomology algebra is now identified with the quotient $\mathbb{C}[p, q]/I$, its spectrum is a subvariety L in the space $\mathbb{C}^{2k}$ with coordinates $(p_1, ..., p_k, q_1, ..., q_k)$ with the ideal $I(L) = I$ (strictly speaking, the variety can be defined only over formal series if the 1-st Chern class c of X is not positive). In any case, it is quasi-homogeneous with $\deg p_i = 1, \deg q_i = D_i$. The space $\mathbb{C}^{2k}$ has the canonical Poisson structure

$$\sum_{i=1}^{k} q_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}$$

which is nothing but extension of the canonical symplectic structure

$$\sum dp_i \wedge \frac{dq_i}{q_i}$$

on the cotangent bundle

$$T^*B = H_2(X, \mathbb{C}) \times [H^2(X, \mathbb{C})/2\pi\sqrt{-1}\mathbb{Z}^k]$$

of the torus B dual to the 2-nd homology lattice $\mathbb{Z}^k$. We claim that the variety L is lagrangian with respect to this symplectic form.

Indeed, interpret the Floer cohomology space

$$FH^*(X, \mathbb{C}) = H^*(X, \mathbb{C}) \otimes \mathbb{C}[q^{\pm 1}]$$

as the space of vector-functions of q with values in the vector space $W = H^*(X, \mathbb{C})$ and introduce the following *operator-valued* 1-form

$$A = \sum A_i(q)\frac{dq_i}{q_i} = (p_1*)\frac{dq_1}{q_1} + ... + (p_k*)\frac{dq_k}{q_k}.$$

Here $A_i = p_i*$ is understood as the operator on W of quantum multiplication by p_i computed at a particular value of q. First of all, we claim that this 1-form satisfies:

$$A \wedge A = 0, \ dA = 0$$

(which means in fact that $\varepsilon d + A\wedge$ is a flat connection operator for all ε). The 1-st identity simply means that the operators A_i commute so as p_i* do. The 2-nd identity means that the matrix elements of A are closed 1-forms and does not follow from any formal properties of quantum multiplication which have been discussed so far. It can be reformulated, in terms of matrix elements of A_i, as follows:

> For any two cycles a and b in X the quantum intersection indices $\langle a|p_i|b\rangle$ are partial derivatives $q_i\frac{\partial S}{\partial q_i}$ of a single (locally defined) function $S = S_{a,b}(q)$.

Put

$$S_{a,b} = \sum_i \langle a, p_i, b\rangle \log(q_i) +$$

$$+ \sum_{\substack{\text{rational curves in } X \\ \text{with } 0 \in a, \infty \in b \text{ of degree } d > 0 \\ \text{and with } c(d) + \dim X = \operatorname{codim}_{\mathbb{C}} a + \operatorname{codim}_{\mathbb{C}} b + 1}} \pm q^d.$$

The 1-st sum is a potential for the constant terms in $\langle a|p_i|b\rangle$ and involves classical intersection indices. The 2-nd sum counts non-constant rational curves, constrained at two points, as if they were discrete. If such a curve contributes by $\pm q^d$ to $S_{a,b}$ then it contributes by $\pm d_i q^d$ to $q_i \partial S_{a,b}/\partial q_i$. Here d_i is exactly the intersection index of a complex hypersurface Poincare-dual to p_i with this rational curve. This means that there are exactly d_i ways to parametrize the curve in such a fasion that $0 \in a$, $\infty \in b$ and $1 \in p_i$, and hence the curve contributes to $\langle a|p_i|b\rangle$ with the same weight $\pm d_i q^d$. This proves our assertion (modulo our usual reservations). In fact this $S_{a,b}$ is one of the "higher order" pairings considered in Conformal Field Theory (actually it is the lower order pairing).

Now the lagrangian property of L follows from a general lemma (which we learned from N.Reshetikhin).

Lemma. *Let*

$$A = \sum_i A_i(t)dt_i$$

be a matrix-valued differential 1-form satisfying $A \wedge A = 0$ and $dA = 0$. Let the scalar differential 1-form

$$p = \sum_i p_i(t)dt_i$$

be its simple *eigen-value. Then p is closed.*

Proof. The assumption actually means that the commuting matrices $A_i(t)$ have a common eigen-vectors $w(t)$ such that $A_i(t)w(t) = p_i(t)w(t)$. Being simple, the eigen-vectors can be chosen smooth in t, and the transposed matrices A_i^* have a smooth field of eigen-covectors $w^*(t)$ (with the same eigen-values) normalized in such a way that $\langle w, w^* \rangle = 1$ identically. Now we have

$$d(pdt) = d(\langle w, w^* \rangle(pdt)) = d\langle Aw, w^* \rangle =$$

$$\langle (dA)w, w^* \rangle - \langle A \wedge dw, w^* \rangle - \langle Aw, dw^* \rangle =$$

$$\langle dw, A^*w^* \rangle - \langle Aw, dw^* \rangle = (d\langle w, w^* \rangle) \wedge (pdt) = 0.$$

Applied to our quantum cohomology situation, this lemma shows that every non-singular local branch of L over B is a lagrangian section of T^*B. This implies that I is a Poisson ideal at least in the case if $I = \sqrt{I}$.

Below we explain how intersection pairings and generating volume functions can be described in terms of geometry on L assuming for simplicity that $I = \sqrt{I}$ and that the 1-st Chern class of X is positive (so that L is indeed a quasi-homogeneous affine algebraic subvariety in $\mathbb{C}^{2k}$ with coordinates (p, q)).

Consider the class in quantum cohomology algebra of $X \times X$ Poincare-dual to the diagonal $X \subset X \times X$. It can be considered as a function on the characteristic lagrangian variety of $X \times X$ which is nothing but $L \times L$. Restrict this function to the diagonal $L \subset L \times L$ and denote the restriction $\Delta \in \mathbb{C}[L]$. Let $\varphi_1, ..., \varphi_N \in \mathbb{C}[L]$ be some quantum cohomology classes. Then for generic $q \in B$

$$\langle \varphi_1|...|\varphi_N \rangle(q) = \sum_{p \in L \cap T_q^*B} \frac{\varphi_1(p)...\varphi_N(p)}{\Delta(p)}$$

and

$$V(z, q) = \sum_{p \in L \cap T_q^*B} \frac{\exp(z_1 p_1 + ...z_k p_k)}{\Delta(p)}.$$

The last remark: since L is lagrangian, the action 1-form on T^*B restricted to L is exact,

$$\sum p_i \frac{dq_i}{q_i}|_L = dC, \quad C \in \mathbb{C}[L].$$

Using quasi-homogeneity of L and Cartan's homotopy formula one can easily show that $C = D_1 p_1 + ... + D_k p_k$ is the 1-st Chern class of X understood as a function on L.

3 Equivariant quantum cohomology

3.1 Why "equivariant"?

In our inductive computation of quantum cohomology of flag manifolds we will encounter the following kind of problems. With a vector bundle over some base B one can associate a fiber bundle $E \to B$ whose fibers are flag manifolds — they consist of flags in the fibers of the vector bundle. Consider the maps of $\mathbb{C}P^1$ with N marked points to E whose composition with the projection to B maps $\mathbb{C}P^1$ to a point and which are holomorphic if considered as maps to the fiber flag manifolds. We will call such holomorphic curves *vertical*.

One may pick N cycles in E and ask how many of such vertical parametrized rational curves of certain homotopy type have the 1-st marked point on the 1-st cycle, the 2-nd marked point — on the 2-nd cycle, and so on.

When the base B is a point, the problem (properly understood of course in terms of intersection indices) becomes a question about structural constants of the quantum cohomology algebra of the flag manifold. Our more general problem about rational curves in flag bundles will not arise in its full generality — we will rather need a sequence of special bundles of flag manifolds over Grassmannians and holomorphic hypersurfaces in the role of the cycles.

On the other hand, this sequence of problems can be understood better in the context of vector bundles over arbitrary finite cellular bases since in such generality it can be replaced by a *universal* problem about the universal vector bundle over the classifying space BG. The total space of the universal flag bundle $E \to BG$ is nothing but the homotopic quotient $EG \times_G F$ of the flag manifold F by the unitary group G. Therefore our universal problem reduces to the question about structural constants of what should be called the *equivariant* quantum cohomology algebra of the flag manifold.

3.2 "Classical" equivariant cohomology

Recall some standard facts [Hs], [AB] about equivariant cohomology.

Let X be a manifold provided with a left action of a compact Lie group G. Consider the *universal* principal G-bundle $EG \to BG$ — a principal G-bundle with contractible total space EG, and define the *homotopic quotient X_G of X by G* as $EG \times_G X = (EG \times X)/G$.

Examples. 1) If X is a point then $X_G = EG/G = BG$.

2) If $H \subset G$ is a Lie subgroup, X is the homogeneous space G/H then $(G/H)_G = EG \times_G (G/H) = (EG \times_G G)/H = EG/H = BH$. For instance, if G is the unitary group U_n and H is its maximal torus T^n so that X is the flag manifold F_n then $X_G = BT^n = (\mathbb{C}P^\infty)^n$.

The equivariant cohomology $H_G^*(X)$ of a G-space X is defined as the ordinary cohomology $H^*(X_G)$ of its homotopic quotient. The natural fibration $X_G \to BG$ (with fiber X), induced by the projection of $EG \times X$ on the first factor, along with Example 1), provide the equivariant cohomology with a module structure over the coefficient algebra $H_G^*(pt)$ of the equivariant theory which is nothing but the characteristic class algebra $H^*(BG)$ of the group G.

Example. For the flag manifold F_n its U_n-equivariant cohomology can be identified with the polynomial ring in n generators $(u_1, ..., u_n)$ since $H^*(\mathbb{C}P^\infty) = \mathbb{C}[u]$ where u is the 1-st Chern

class of the universal Hopf circle bundle. The module structure over the algebra of universal Chern classes $H^*(BU_n) = \mathbb{C}[c_1, ..., c_n]$ becomes more "visible" if we represent the equivariant cohomology of the flag manifold as the quotient of the polynomial algebra $\mathbb{C}[u, c]$ by the ideal of relations $c_i = \sigma_i(u)$, $i = 1, ..., n$, where σ_i are elementary symmetric polynomials of $(u_1, ..., u_n)$.

Similarly, equivariant cohomology of cartesian products of flag manifolds are tensor products of equivariant cohomology of factors and they are modules over characteristic class algebras of products of unitary groups. Of course, this is a general property of products ΠX_i of G_i-spaces.

3.3   Equivariant intersection indices

Consider a D-dimensional compact oriented G-manifold X and the associate X-bundle $\pi : X_G \to BG$. Since we are actually going to apply our general constructions to homogeneous complex manifolds it is convenient to make a convention right now that all the dimensions are *complex* ones, and therefore dimensions of real manifolds or cycles can be *half*-integral. With this convention in force, let us consider equivariant cohomology classes $p_1, ..., p_N$ of X of total degree M and define their *intersection index* $\langle p_1, ..., p_N \rangle$ with values in the structural ring $H_G^*(pt)$ of equivariant theory.

If C is a homology class of BG of degree K one can construct its *inverse image* $\pi^{-1}(C)$ which is geometrically the preimage of the cycle C in the bundle $\pi : X_G \to BG$ and represents a homology class of degree $K + D$ in X_G. By definition,

$$\langle p_1, ..., p_N \rangle[C] = (p_1...p_N)[\pi^{-1}(C)].$$

This formula describes the intersection cohomology class through its evaluation on homology classes and may give rise to a non-zero result only if $M = K + D$ of course. In the case when an infinite-dimensional manifold has been chosen on the role of the classifying space BG one may also think of $p_1, ..., p_n$ as cycles of finite total *co*dimension M, and of $\langle ... \rangle[C]$ as the mutual intersection index of $p_1, ..., p_N$ and $\pi^{-1}(C)$. In the case if C is a point our definition reduces to the ordinary intersection index in X of cycles Poincare-dual to the restrictions of the cohomology classes p_i to the fiber of π.

The equivariant intersection indices $H_G^*(X)^{\otimes N} \to H_G^*(pt)$ have the following more or less obvious properties:

1. They are homogeneous of degree $-\dim X$ (with our convention in force);

2. They are $H^*(pt)$-multi-linear;

3. They are totally anti-symmetric (notice that $H^*(pt)$ happened to be commutative);

4. They are determined by cup-multiplication in $H_G^*(X)$ and by the "intersection index" $H_G^*(X) \to H_G^*(pt)$ with $N = 1$ which is nothing but the *direct image* operation $\pi_! : H^*(X_G) \to H^*(BG)$ dual to the inverse image in homology.

In terms of differential forms the direct image operation consists in fiberwise integration.

Our objective for the moment is to describe explicitly the direct image for equivariant cohomology of flag manifolds.

Proposition. *For the flag manifold F_n the direct image $\pi_! : \mathbb{C}[u] \to \mathbb{C}[c]$ is given by the following Cauchy formula:*

$$(\pi_! f)(c) = (\frac{1}{2\pi i})^n \int_{T^n} \frac{f(u)du_1 \wedge ... \wedge du_n}{(\sigma_1(u) - c_1)...(\sigma_n(u) - c_n)}.$$

The integral equals the total sum of residues in $\mathbb{C}^n$. In other words, in order to find the direct image of a polynomial $f(u)$ one first constructs its total alternation

$$\text{Alt} f(u) = \sum_{w \in S_n} (-1)^{\varepsilon(w)} f(wu),$$

then divides it by the "fundamental anti-invariant" (= Vandermond)

$$\Delta_n(u) = \det(\frac{\partial \sigma_i(u)}{\partial u_j})$$

and expresses the ratio $\text{Alt} f/\Delta_n$ as a polynomial $\hat{f}(\sigma(u))$ of elementary symmetric functions: $\hat{f}(c_1, ..., c_n)$ is then the direct image of f.

The main argument in the proof of this formula is "what else can it be?"

Indeed, due to linearity property the direct image operation is completely determined by its action on generators of $\mathbb{C}[u]$ as a $\mathbb{C}[c]$-module. The generators can be chosen as homogeneous representatives of a linear basis in the ordinary cohomology $\mathbb{C}[u]/(\sigma_1(u), ..., \sigma_n(u))$ of the flag manifold (Nakayama lemma!). Due to the degree reasons these representatives all have zero direct images except the generator Poincare dual to the fundamental cycle. The latter has constant direct image, and the constant can be easily found equal 1 (evaluate the direct image at a point). The residue formula (and the operation $\Delta_n^{-1}\text{Alt}$) do have all there properties since $\deg \Delta_n$ "accidentally" equals $\dim F_n$.

One more example. Consider the subgroup $G' = U_m \times U_{n-m} \subset U_n = G$ and the bundle $BG' \to BG$ with the fiber $G/G' = Gr(n, m)$. The direct image operation

$$\text{Direct image}: \ H^*(BG') = \mathbb{Z}[c'_1, ..., c'_m, c''_1, ..., c''_{n-m}] \to \mathbb{Z}[c_1, ..., c_n] = H^*(BG)$$

in this bundle somehow transforms partially symmetric polynomials of $(u', u'') = ((u_1, ..., u_m), (u_{m+1}, ..., u_n))$ to totally symmetric ones, since

$$c'_i = \sigma_i(u'), \ c''_j = \sigma_j(u''), \ c_r = \sum_{i=0}^{r} \sigma_i(u')\sigma_{r-i}(u'') = \sigma_r(u)$$

(where $\sigma_0 = 1$).

Corollary.

$$[Direct\ image\ f](\sigma(u)) = \frac{Alt[\Delta_m(u')\Delta_{n-m}(u'')f(\sigma(u'), \sigma(u''))]}{m!(n-m)!\Delta_n(u)}.$$

Proof. We can represent $f(c', c'')$ as the direct image $\Pi_! g(u', u'')$ of some $g(u)$ in the product of bundles $\Pi : BT^m \times BT^{n-m} \to BU_m \times BU_{n-m}$ and thus identify $[Direct\ image\ f]$ with $\pi_! g$.

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3.4   Instanton corrections

Let X be a complex Kahler manifold of dimension D provided with a holomorphic action of the complexified compact Lie group $G \subset G_{\mathbb{C}}$. We will assume for simplicity that X that $H^{1,1}(X) = H^2(X)$. Notice that the lattice $\mathbb{Z}^k$ is a sublattice in the second homology group of the homotopic quotient X_G and thus classes of *vertical* rational curves in the total space of the bundle $X_G \to BG$ are canonically identified with elements of $\mathbb{Z}^k$.

We define *quantum equivariant intersection indices* as follows.

Let $p_1, ..., p_N$ be cycles in X_G of finite codimensions which add up to M. Their quantum intersection index $\langle p_1|...|p_N \rangle$ will be an element of the algebra $H_G^*(pt, \mathbb{Z}[[q]])$. Given a K-dimensional cycle $C \subset BG$, we define the value $\langle p_1|...|p_N \rangle[C]$ as the sum of contributions of rational parametrized curves $\varphi : \mathbb{C}P^1 \to \pi^{-1}(C)$ in the *fibers* of the bundle $\pi : X_G \to BG$ restricted to C such that N marked points $x_1, ..., x_N$ in $\mathbb{C}P^1$ map to the cycles $p_1, ..., p_N$ respectively: $\varphi(x_i) \in p_i$. The contribution of φ is non-zero only if $c(d) + D + K = M$ and equals $\pm q^d$ in the assumptions of course that the cycles p_i are in general position with respect to the family of vertical rational curves φ of degree d, that the family indeed has the dimension $c(d) + D + K$ predicted by the Riemann–Roch formula, and that the contributing curves are regular points in this family:

$$\langle p_1|...|p_N \rangle[C] = \sum_{\substack{\text{vertical discrete holomorphic maps:} \\ (\mathbb{C}P^1, x_1, ..., x_N) \to (\pi^{-1}(C), p_1, ..., p_N) \\ \text{of degree } d}} \pm q^d.$$

The sign $\pm$ in this formula can be defined naturally in terms of intersection indices in moduli space; it is "plus" at least in the case if all the cycles p_i and C are holomorphic (the latter assumes that a complex manifold is taken on the role of BG), and will be described in 3.5 for arbitrary C.

Rigorous justification of this construction, and in particular — verification that the intersection indices actually depend only on the (co)homology classes represented by the cycles p_i and C, encounters the same difficulties as in the case of the quantum non-equivariant intersection indices. In particular, bringing to general position may involve perturbations of the complex structure towards almost complex ones which in our case should be done fiberwise in the bundle $X_G \to BG$ and do not have to be the same on all fibers.

Intersection indices $\langle \ |...| \ \rangle$ have the following obvious properties relating them with "classical" intersection indices $\langle \ , ..., \ \rangle$:

1. they are multi-linear and skew-symmetric;

2. $\langle p_1|...|p_N|[1] \rangle = \langle p_1|...|p_N \rangle$, where $[1]$ represents the fundamental cycle in X_G;

3. $\langle p_1|...|p_N \rangle|_{q=0} = \langle p_1, ..., p_n \rangle$ — they are q-deformations of classical intersection indices;

4. $\langle p_1|p_2 \rangle = \langle p_1, p_2 \rangle$ so that $\langle p|[1] \rangle$ coincides with the classical direct image operation;

   and a less obvious

5. $H^*_G(pt)$-multi-linearity property (where '·' stands for the cap-product, Poincare dual to the ordinary multiplication of cohomology classes represented by finite codimension cycles)

$$\langle \pi^*(p) \cdot p_1|...\rangle[C] = \langle p_1|...\rangle[p \cap C] = (p \cdot \langle p_1|...\rangle)[C]$$

which means that a *vertical* rational curve in X_G which has a common point with the preimage $\pi^{-1}(p)$ of a finite codimension cycle $p \subset BG$ in the base, is entirely contained in this preimage.

Similarly to ordinary quantum cohomology, quantum equivariant intersection indices have a few other interpretations.

3.5 Intersections in 'moduli spaces'

Consider the product X^N of N copies of X as a G-manifold provided with the diagonal G-action. The homotopic quotient X^N_G has N canonical projections $X^N_G \to X_G$ compatible with the projections $X^N_G \to BG$, $X_G \to BG$ to the classifying space. Let $p_1, ..., p_N$ be equivariant cohomology classes of X. One may think of them as represented by finite codimension cycles in X_G, one in each of N copies. Pulled back to X^N_G they define N equivariant cohomology classes of X^N which we denote $p_1, ..., p_N$ too.

Let $\mathcal{M}_d$ denote the space of parametrized rational curves $\varphi : \mathbb{C}P^1 \to X$ of certain degree (= homology class) d. Evaluation map $\mathcal{M}_d \to X^N$, $\varphi \mapsto \varphi(x_1), ..., \varphi_N(x_N)$ at N generic points in $\mathbb{C}P^1$ defines a G-invariant complex subvariety in X^N. Its fundamental cycle $\bar{\mathcal{M}}_d$ determines an equivariant cohomology class of X^N: it is Poincare-dual to

$$EG \times_G \bar{\mathcal{M}}_d \subset EG \times_G X^N.$$

We denote this equivariant class $[\mathcal{M}_d]$.

One defines the quantum equivariant intersection index using classical equivariant indices in $H^*_G(X^N)$ as

$$\langle p_1|...|p_N\rangle = \sum_d \langle p_1, ..., p_N, [\mathcal{M}_d]\rangle q^d.$$

It is easy to see what is the meaning of the RHS, evaluated at a cycle $C \subset BG$: it counts the numbers of discrete rational maps φ to the fibers of the bundle $\pi^{-1}(C) \to C$ such that $\varphi(x_i)$ is in the cycle representing p_i in X_G. The maps are "weighted" by the factors q^d and are counted with the signs prescribed by (co)orientations of the cycles. In particular, this construction (being at least morally equivalent to the first one) specifies how the signs $\pm$ in the previous definition should be chosen.

3.6 Integrals in loop spaces

The quantum intersection indices defined by means of evaluation maps are (expected to be) independent on the choice of evaluation points $x_1, ..., x_N$ on the projective line provided that the points are generic (and in particular distinct). Therefore one can replace $\langle p_1, ..., p_N, [\mathcal{M}_d(x)]\rangle$ by its average value

$$\int_{T^N} \langle p_1, ..., p_N, [\mathcal{M}_d(x)]\rangle dx_1...dx_N$$

where T^N is a torus in the configuration space $(\mathbb{C}P^1)^N$ of N points $x = (x_1, ..., x_N)$, namely the product of N standard unit circles in $\mathbb{C}P^1 = \mathbb{C} \cap \infty$ (notice that T^N is dense in Zarissky topology on $(\mathbb{C}P^1)^N$). This formula allows us to interpret the intersection indices as some integrals of differential forms on loop spaces.

Suppose that the classifying space BG is chosen in the form of infinite-dimensional manifold and that the equivariant cohomology classes $p_1, ..., p_N$ are represented by closed differential forms on X_G. Such a differential form determines a differential form of the same degree on the space of free loops in X_G. Namely, if $t \mapsto \gamma(t)$ is a loop, the average $\oint p_t dt$ is an exterior form on the space of vector fields along the loop, and thus $P = \oint p_d t$ is a differential form on the loop space, closed if p is closed on X_G.

Furthermore, we interpret a (vertical) rational curve $\varphi : \mathbb{C}P^1 \to X_G$ as an "algebraic loop" restricting the map φ to the unit circle $T \subset \mathbb{C} - 0 \subset \mathbb{C}P^1$. Now on we may think of the spaces $\mathcal{M}_d$ of rational maps, as well as of the spaces $\mathcal{M}_d[C]$ of such vertical rational maps to the fibers of the bundle $X_G \to BG$ over a given cycle $C \subset BG$, as subsets (chains, cycles) in the loop space.

The above integral over the torus immediately turns into the integral in the loop space,

$$\langle p_1, ..., p_N, [\mathcal{M}_d] \rangle [C] = \int_{\mathcal{M}_d[C]} P_1 \wedge ... \wedge P_N.$$

As usual, this formula assumes that the integral equals zero unless the total degree M of the wedge product equals the dimension $c(d) + D + K$ of the chain $\mathcal{M}_d[C]$.

We will make use of this construction in the special case when the equivariant cohomology algebra $H_G^*(X)$ is generated (as algebra) by the classes of degree 2 — that is of degree 1 taking into account our convention that all the dimensions and degrees are complex. Let $p_1, ..., p_n$ now denote a set of such generators, i. e. a basis in $H_G^2(X)$. We prefer to think of p_i as of closed differential 2-forms on the infinite-dimensional manifold X_G, or even as of symplectic (or Kahler) forms, taking into account our assumptions about X and the fact that classifying spaces of compact Lie groups have Kahler models. Denote

$$P(z) = z_1 P_1 + ... + z_n P_n$$

a general linear combination of the differential (symplectic, Kahler) 2-forms P_i on the loop space of X_G corresponding to the forms p_i on X_G. Let us define the *generating volume function* $V \in H_G^*(pt, \mathbb{Z}[[z, q]])$ — a formal series in q and z with coefficients in the ring of characteristic classes, such that the value of V on a homology class represented by the cycle $C \subset BG$ is equal to the weighted oriented volume

$$V|_{[C]} = \sum_d q^d \int_{\mathcal{M}_d[C]} \exp(P(z))$$

of the space $\mathcal{M}[C] = \cup_d q^d \mathcal{M}_d[C]$ of vertical rational curves over C. Here $\exp(P)$ stands for

$$\sum_{k=0}^{\infty} \frac{1}{k!} P \wedge ... \wedge P \ (k \text{ times})$$

so that the integral $\int_{\mathcal{M}} \exp(P)$ really represents the symplectic k-dimensional volume of a k-cycle $\mathcal{M}$ if the form P is symplectic (we should notice however that orientation of C contributes the sign of the "volume").

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The generating volume function has not so many non-zero terms as one could think: due to dimension reasons it is *weighted-homogeneous of degree* $-D$ when the degrees of the variables are assigned as

$$\deg q^d = c(d), \ \deg z_i = -1$$

and characteristic classes from $H^*_G(pt)$ have their natural degrees.

One of applications of this function describes quantum intersection indices of the generators p_i:

$$\langle p_{i_1}|...|p_{i_N}\rangle = \frac{\partial^N}{\partial z_{i_1}...\partial z_{i_N}}|_{z=0} \ V(z)$$

(it is just the property of the exponential series).

Another property of the volume generating functions, that we are going to exploit, is their simple behavior under *product, restriction* and *induction* operations.

Product. Let X', X'' be compact Kahler G'- and G''-spaces respectively, and $V'(z', q') \in H^*_{G'}(pt)$, $V''(z'', q'') \in H^*_{G''}(pt)$ be the corresponding generating volume functions. Then the generating volume function V for the $G' \times G''$-space $X' \times X''$ is

$$V((z', z''), (q', q'')) = V'(z', q')V''(z'', q'').$$

Indeed, the homotopic quotient of $X' \times X''$ is the product of $X'_{G'}$ and $X''_{G''}$ fibered over the product $BG' \times BG''$ of classifying spaces. A holomorphic map to $X' \times X''$ is a pair of holomorphic maps to X' and X'' respectively and hence the chain $\mathcal{M}_{d',d''}$ factors:

$$\mathcal{M}_{d',d''}[C' \times C''] = \mathcal{M}_{d'}[C'] \times \mathcal{M}_{d''}[C''].$$

Its volume with respect to $P(z) = P'(z') \oplus P''(z'')$ is the product of corresponding volumes and therefore

$$\sum_{(d',d'')} (q')^{d'} (q'')^{d''} \int_{\mathcal{M}_{d'}[C'] \times \mathcal{M}''_d[C'']} \exp(P(z)) =$$

$$[\sum_{d'}(q')^{d'} \int_{\mathcal{M}_{d'}[C']} \exp(P'(z'))] \cdot [\sum_{d''}(q'')^{d''} \int_{\mathcal{M}''_d[C'']} \exp(P''(z''))].$$

Restriction. Let X be a compact Kahler G-space and $G' \subset G$ be a Lie subgroup. Considering X as a G'-space, we obtain an X-bundle $X_{G'} \to BG'$ (induced, as a bundle, from $X_G \to BG$ by means of the natural map $\pi : BG' \to BG$ of classifying spaces) and the corresponding map of total spaces $\zeta : X_{G'} \to X_G$ with the fiber G/G'. Then for the generating volume functions $V(z, q)$ and $V'(z', q)$ we have

$$V'(\zeta^*(z), q) = \pi^* V(z, q).$$

Indeed, for a cycle $C' \subset BG'$ the bundle $\mathcal{M}_d[C'] \to C'$ is induced by π from $\mathcal{M}_d[\pi_* C'] \to \pi(C')$ and therefore

$$\int_{\mathcal{M}_d[C']} \exp(\zeta^*(P(z)) = \int_{\mathcal{M}_d[\pi_* C']} \exp(P(z)).$$

In particular, if G' is trivial so that π is $EG \to BG$ and $X_{G'} = EG \times X$, then the homomorphism $\zeta^* : H^2(X_G) \to H^2(X)$, $z \mapsto z'$, is onto, and the generating volume function $V'(z', q)$ coincides with the non-equivariant one and can be computed from $V(z, q)$ as its reduction $H^*_G(pt) \to \mathbb{Z}$ modulo G-characteristic classes of positive degree.

This implies that non-equivariant quantum intersection indices $\langle p_{i_1}|...|p_{i_N} \rangle$ are obtained by such a reduction from the corresponding quantum equivariant intersection indices.

Induction. Let $G' \subset G$ be a subgroup with a simply-connected compact Kahler quotient G/G', and Y be a compact Kahler G'-space. We construct a compact Kahler G-space $X = G \times_{G'} Y$ and call it *induced* from Y (like induced representations). In fact X is fibered over G/G' with the fiber Y. The homotopic quotient spaces of X and Y coincide:

$$X_G = EG \times_G (G \times_{G'} Y) = EG \times_{G'} Y = Y_{G'},$$

and thus their equivariant cohomology is the same, but the module structure in $H^*_G(X)$ is induced from the module structure in $H^*_{G'}(Y)$ by the natural map $BG' \to BG$.

Let p'' be a basis of non-negative classes in $H^2(G/G')$ lifted to X, and $p = (p', p'')$ be its extension to such a basis in $H^2(X)$. Encoding the homology class of a rational curve in X by the string $(d', d'') = (d_1, ..., d_k)$ of its degrees with respect to the dual basis in $H_2(X)$, we find that the curves vertical in the bundle $X \to G/G'$ have $d'' = 0$ and *vice versa*.

This means that the quantum deformation ring $\mathbb{Z}[q']$ for Y can be considered as a *quotient* of the corresponding ring for X:

$$\mathbb{Z}[q'] = \mathbb{Z}[q', q'']/(q'').$$

Remark. This identification may seem confusing, since the group algebra $\mathbb{C}[q'^{\pm 1}]$ is a *subalgebra* in $\mathbb{C}[q^{\pm 1}]$. In fact, replacing the algebra $\mathbb{C}[q^{\pm 1}]$ of functions on the torus by the polynomial algebra $\mathbb{C}[q]$ defines, in geometrical terms, partial compactification of the torus to $\mathbb{C}^k$. Our description of $\mathbb{C}[q']$ as a quotient corresponds to the embedding of such a compactified torus $\mathbb{C}^{k'}$ for Y into the "boundary" $\mathbb{C}^k - (\mathbb{C} - 0)^k$ of the torus for X.

Denote $V'(z, q')$ and $V(z, q)$ the generating volume functions for quantum equivariant cohomology of Y and X respectively. Then

$$V(z, (q', 0)) = \text{Direct image } V'(z, q')$$

where the direct image operation refers to the bundle $\pi : BG' \to BG$.

Indeed, when we evaluate $V(z, q)$ on some cycle $C \subset BG$ at $q'' = 0$ we simply calculate weighted volume of the space of vertical algebraic loops in X_G over C but throw away contributions of all rational curves with $d'' \neq 0$. But a rational curve in X with $d'' = 0$ projects to G/G' to a point. This means that the LHS actually computes weighted volume of the space of vertical algebraic loops in $Y_{G'} \to BG' \to BG$ over the preimage $C' = \pi^{-1}C$. Therefore

$$V(z, (q', 0))[C] = V'(z, q')[C'] = [\text{Direct image } V'(z, q')][C]$$

by the very definition of the direct image operation.

3.7 Equivariant Floer homology

We briefly discuss here quantum equivariant cohomology from the point of view of Morse-Floer theory on loop spaces. This discussion is supposed to motivate our conjecture that the general properties expected from quantum cohomology can be naturally generalized to the equivariant case.

Let X, as above, be a compact simply-connected Kahler manifold provided with a holomorphic action of the complexified compact Lie group $G_{\mathbb{C}}$ and and with a G-invariant Kahler form. The group $G_{\mathbb{C}}$ also acts by holomorphic transformations on the loop space LX and its universal covering. Since the action functional $\mathcal{A}$ on the covering is G-invariant one can try to construct the equivariant Floer (co)homology $FH_G^*(X)$ by means of equivariant Morse-Witten theory for $\mathcal{A}$.

Usually one defines an equivariant Morse chain complex using finite-dimensional approximations $EG_N \to BG_N$ of the universal G-bundle. For example, if G is the unitary group U_n one can choose the complex Grassmann manifold $Gr(N, n)$ on the role of BG and the corresponding Stiefel manifold on the role of EG_N. Mimicking this approach, we can extend the functional $\mathcal{A}$ to the space $EG_N \times LX$ in the trivial manner and thus construct a functional $\mathcal{A}_N$ on the manifold $L_N = EG_N \times_G LX$ approximating the homotopic quotient space $(LX)_G$. Now we can apply Floer's semi-infinite Morse theory to the functionals $\mathcal{A}_N$. Notice that the homotopic quotient $(LX)_G$ is nothing but the space of vertical loops in the bundle $X_G \to BG$, and L_N is simply its restriction to $BG_N \subset BG$.

Taking care of the riemannian metric, add a G-invariant riemannian metric on EG_N as a direct summand to the Kahler G-invariant metric on LX induced from that on X. Then the gradient vector field of $\mathcal{A}$ on $EG_N \times LX$ is tangent to the second factor and is invariant with respect to the diagonal action of G. This means that the gradient vector field of $\mathcal{A}_N$ relative to the factor-metric on L_N is just the projection of that G-invariant field, and the corresponding gradient flow consists in *fiberwise* analytic continuation of vertical loops in the X-bundle $L_N \to BG_N$. In particular, Floer cohomology of L_N will carry a module structure over the ordinary cohomology algebra of BG_N.

Notice that the G-action on LX commutes with both the circle action (= reparametrization of loops) and the action of the covering transformation group $\mathbb{Z}^k$ (so that both actions survive on L_N). The first implies that $\mathcal{A}_N$ is a perfect Morse-Bott function on L_N (see [G],[G1]). The second describes the action of the group ring $\mathbb{Z}[q^{\pm 1}]$ on the Floer cohomology of L_N, which is therefore additively isomorphic to the cohomology $H^*((X_G)_N, \mathbb{Z}[q^{\pm 1}])$ of the critical point set.

Passing to the limit $N \to \infty$, we conclude that G-equivariant Floer cohomology $FH_G^*(X)$ of LX should be a $H_G^*(pt, \mathbb{Z}[q^{\pm 1}])$-module canonically isomorphic to the equivariant cohomology of X with coefficients in the group ring $\mathbb{Z}[q^{\pm 1}]$.

A multiplicative structure in equivariant quantum cohomology of LX can be defined by means of the evaluation map at the point $1 \in S^1$:

$$LX \to (LX \times X), \ \ (\gamma : S^1 \to X) \mapsto (\gamma, \gamma(1)).$$

This map is G-equivariant and induces an action of equivariant cohomology classes of X by module endomorphisms on equivariant Floer cohomology $FH_G^*(X)$ of the loop space LX . Using

our explicit description of the gradient flow on $(LX)_G$ as fiberwise analytic continuation of loops, one can compute this action in terms of vertical holomorphic curves and quantum equivariant intersection indices $\langle |...| \rangle$ introduced in 3.3. Namely the action of $p \in H_G^*(X)$ on $a \in FH_G^*(X)$ satisfies

$$\langle p * a, b \rangle = \langle a|p|b \rangle$$

for any $b \in FH_G^*(X)$ where the pairing on the LHS is the classical equivariant intersection index on $H_G^*(X, \mathbb{Z}[q^{\pm 1}])$ with values in $H_G^*(pt, \mathbb{Z}[q])$.

The multiple quantum equivariant intersection indices $\langle a|p_1|...|p_r|b \rangle$ can be expressed in a similar manner in terms of evaluation maps $LX \to LX \times X^r$ at r distinct points $x_1, ..., x_r$ on the circle S^1. We conjecture that they satisfy the "principal axiom" of Topological Field Theory (see 2.2). This conjecture implies that the multiple intersection indices represent matrix elements of compositions of the endomorphisms corresponding to $p_1, ..., p_r \in H_G^*(X)$. Finally, if one defines quantum equivariant cohomology of $QH^*(X)$ as the algebra generated by these endomorphisms and operators of multiplication by q, then our conjecture means that this algebra

- is additively isomorphic to $H_G^*(X, \mathbb{Z}[q])$ (or may be "$[[q]]$"),

- provides a "quantum" deformation of the classical equivariant cohomology algebra $H_G^*(X)$,

- inherits the module structure over $H_G^*(pt) \otimes \mathbb{Z}[q]$, and

- allows to express the multiple pairings through quantum multiplication and the classical direct image functional:
$$\langle p_1|...|p_r \rangle = \langle p_1...p_r, [1] \rangle.$$

It is difficult to say now whether a rigorous justification of these hypotheses should be even more sophisticated than in the non-equivariant case. One one hand, general position arguments should require introducing almost complex structures on X which are not G-invariant. The most natural way to handle this problem — by considering the space $\mathcal{J}$ of all almost complex structures and constructing G-equivariant Floer cohomology of $LX \times \mathcal{J}$ — involves one more "infinity" and seems to raise the level of technical difficulty. On the other hand, the finite-dimensional approximations BG_N of classifying spaces have Kahler models, and quantum equivariant cohomology of X seem to be expressible in terms of non-equivariant quantum cohomology of the approximations $(X_G)_N \to BG_N$: it suffices to "throw away" contributions of non-vertical rational curves in $(X_G)_N$, i. e. put some of 'q's equal zero. This approach can possibly reduce the problem back to the axioms of non-equivariant Topological Field Theory.

We are not ready to discuss further this problem here. We also leave for the reader to think out the parallel construction of equivariant quantum multiplication which is based on composition of loops.

3.8 Characteristic classes as Casimir functions

Here we interpret the quantum equivariant cohomology algebra $QH_G^*(X)$ as the algebra of functions on some lagrangian variety in the assumption that the ordinary cohomology algebra $H^*(X)$ of the simply-connected Kahler manifold X is generated by non-negative $(1,1)$-classes $p_1, ..., p_k$ (in

notations of 2.3). This assumption along with the spectral sequence of the X-bundle $X_G \to BG$ implies that the equivariant cohomology algebra $H_G^*(X)$ is additively isomorphic to $H^*(BG) \otimes H^*(X)$ and is generated, as an $H_G^*(pt)$-algebra, by k elements representing $1 \otimes p_i$ which we will denote $p_1, ..., p_k$ again.

Its quantum deformation $QH_G^*(X)$ has been defined in 3.7 by means of the identity

$$\langle a * b, c \rangle = \langle a | b | c \rangle.$$

Considered as $H_G^*(pt)$-algebra, it is generated by $(p_1, ..., p_k, q_1, ..., q_k)$ and is therefore isomorphic to the quotient of the polynomial algebra $H_G^*(pt)[p, q]]$ by some ideal of relations.

Passing to complex coefficients and introducing temporary notations c_i, $i = 1, ..., r$, for generators of the polynomial algebra $H^*(pt, \mathbb{C}) = \mathbb{C}[c]$ of G-characteristic classes, we interpret the quantum equivariant cohomology algebra $QH_G^*(X, \mathbb{C})$ as the algebra of regular functions on a (quasi-homogeneous) subvariety $\mathcal{L}$ determined by the ideal of relations $\mathcal{I}$ in the complex space with coordinates

$$(p_1, ..., p_k, q_1, ..., q_k, c_1, ..., c_r).$$

This complex space has a natural Poisson structure

$$q_1 \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + ... + q_k \frac{\partial}{\partial q_k} \wedge \frac{\partial}{\partial p_k}$$

due to the constant coefficient pairing between $H^2(X) = H^2(X_G)/H^2(BG)$ and $\mathbb{Z}^k = H_2(X) \subset H_2(X_G)$ (we assume of course that the basis in the lattice $\mathbb{Z}^k$ is dual to the basis $(p_1, ..., p_k)$ in $H^2(X)$).

We observe that the characteristic classes c_i play the role of Casimir functions of such a Poisson structure and claim that the characteristic variety $\mathcal{L}$ is lagrangian in the sense of Poisson geometry, i. e. its intersections with the symplectic leaves $\vec{c} = const, q_1...q_k \neq 0$ are lagrangian at their regular points.

Similarly to the non-equivariant case 2.3, this statement is based on the properties of the matrix-valued differential 1-form $A = \sum (p_i *)(dq_i)/q_i$ to satisfy $dA = 0, A \wedge A = 0$, but now the Casimir functions c_i are treated by the differential d and by the operators $p_i *$ as constants. Mimicking 2.3, we introduce a $\mathbb{C}[[c, q]]$-valued bilinear form on $H_G^*(X, \mathbb{C}[[q]])$ by the formula

$$S_{a,b}|_{[C]} = \sum_{\substack{\text{degree } d \text{ isolated vertical rational curves} \\ \text{in } X_G \to BG \text{ restricted to } C \subset BG \\ \text{with two marked points in } a \text{ and } b}} \pm q^d$$

which evaluates the bilinear form of two finite codimension cycles $a, b \subset X_G$ on a finite-dimensional cycle $C \subset BG$.

Thinking of c_i as of the preimage in X_G of a finite-codimension cycle in BG we immediately conclude that S is $\mathbb{C}[c]$-bilinear:

$$S_{c_i a, b}|_{[C]} = S_{a,b}|_{[c_i \cap C]} = (c_i S_{a,b})|_{[C]}.$$

Thinking of p_i as a complex hypersurface in X_G we find, as in 2.3, that

$$S_{a,b} + \sum \langle a, p_i, b \rangle \log(q_i)$$

is a potential for the (a, b)-matrix element of the 1-form A:

$$q_i \frac{\partial}{\partial q_i} S_{a,b} = \langle a | p_i | b \rangle - \langle a, p_i, b \rangle.$$

This is equivalent to $dA = 0$ and together with commutativity $A \wedge A = 0$ and the lemma in 2.3 implies that each non-singular branch of $\mathcal{L} \cap \{\vec{c} = const\}$ over the torus with coordinates q is lagrangian in the cotangent bundle of this torus (= the symplectic leaf with coordinates p, q).

4   Computation of $QH^*_{U_n}(F_n)$

In this section, we compute quantum (equivariant) cohomology of flag manifolds. The results here are mathematically rigorous *corollaries* of the following *conjectures* about general properties of quantum cohomology of Kahler manifolds:

- Quantum equivariant cohomology is a skew-commutative associative algebra over the characteristic class ring;

- It is a weighted-homogeneous q-deformation of the classical equivariant cohomology;

- Equivariant generating volume functions satisfy the product, restriction and induction properties from 3.6.

4.1   Root systems

The structure of the 2-nd (co)homology lattice of flag manifolds can be understood better in terms of root systems. The flag manifold F_n is the space $G_{\mathbb{C}}/B$ of all Borel subalgebras in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_n(\mathbb{C})$. Therefore its tangent bundle splits canonically into the direct sum of line bundles $\oplus_\alpha L_\alpha$ indexed by *positive roots* α of the root system A_{n-1}. Recall that this root system can be described as the set of linear functions $x_i - x_j$ on the lattice $\mathbb{Z}^n$ with coordinates $x_1, ..., x_n$, and the positive roots are those with $i < j$. The $n-1$-dimensional lattice spanned by the roots can be identified with a finite index sublattice in the 2-nd cohomology group $H^2(F_n)$ by the map

a line bundle $\mapsto$ its 1-st Chern class.

Therefore the 1-st Chern class of the flag manifold is represented by the total sum 2ρ of positive roots. According to Borel-Weil theory, finite-dimensional representations of SU_n can be realized in spaces of holomorphic sections of non-negative line bundles over F_n and correspond in a $1-1$ fashion to their 1-st Chern classes. This theory implies that the Kahler cone of F_n is the Weyl chamber spanned by the 1-st Chern classes $p_1, ..., p_{n-1}$ of the *fundamental* line bundles $\det^* \Lambda^i \mathbb{C}^n, i = 1, ..., n-1$, called — in terms of the root system — *fundamental weights*.

The fundamental weights $p_i = x_1 + ... + x_i, i = 1, ..., n-1$, form a basis in the lattice $H^2(F_n)$. The vectors $\alpha_1, ..., \alpha_{n-1}$ of the dual basis and their non-negative integer combinations represent, in

the homology group $H_2(F_n)$, classes of *holomorphic* curves in F_n. Identifying the space $H^*(F_n, \mathbb{Q})$ with its dual by means of the Weyl-invariant inner product (the Cartan matrix is its matrix in the basis of fundamental weights) we find that $(\alpha_1, ..., \alpha_{n-1})$ becomes the basis of *simple* roots $\alpha_i = x_i - x_{i+1}$ under this identification.

Now the famous identity

$$\sum_{\alpha > 0} \alpha = 2\rho = 2(p_1 + ... + p_{n-1})$$

along with $\langle p_i, \alpha_j \rangle = \delta_{ij}$ means that in our representation of classes $\sum d_i \alpha_i$ of rational curves by monomials $q_1^{d_1}...q_{n-1}^{d_{n-1}}$ the degrees of the variables q_i are

$$\deg q_i = c(\alpha_i) = \langle 2\rho, \alpha_i \rangle = 2.$$

4.2 Auxiliary bundle

According to general theory,

$$QH^*_{U_N}(F_n) = \mathbb{Z}[u_1, ..., u_n, q_1, ..., q_{n-1}, c_1, ..., c_n]/I_{U_n}$$

where the ideal I_{U_n} is generated by some quasi-homogeneous q-deformation of the relations

$$c_i = \sigma_i(u), i = 1, ..., n, \quad \deg u_i = 1, \deg c_i = i, \deg q_i = 2$$

which can be written (using a formal variable λ of degree 1) as a single quasi-homogeneous identity of degree n:

$$(u_1 + \lambda)...(u_n + \lambda) = \lambda^n + \sigma_1 \lambda^{n-1} + ... + \sigma_n.$$

We find this deformation by induction on $n = 2, 3, 4...$, based on the following obvious

Lemma 1. *For $n > 2$, suppose that a quasi-homogeneous relation of the form*

$$(u_0 + \lambda)...(u_n + \lambda) - [\lambda^n + \sigma_1 \lambda^{n-1} + ... + \sigma_n] = O(q_1, ..., q_{n-1})[\lambda, q, u, \sigma]$$

is satisfied in quantum equivariant cohomology algebra of the flag manifold F_n modulo q_i for each $i = 1, ..., n-1$. Then this relation holds identically (i. e. for all q).

Proof. Indeed, since the LHS of the relation in question is homogeneous of degree n, the hypothesis of Lemma 1 means that the difference $LHS - RHS$ is divisible by $q_1...q_{n-1}$. But $\deg q_i = 2$ and

$$\deg q_1...q_{n-1} = 2n - 2 > n \text{ for } n > 2.$$

This implies that $LHS - RHS = 0$.

Remark. This lemma is the only place in our proof where we use some specificity of the group U_n. It also holds for flag manifolds of series C and D but fails for other compact simple Lie groups. For their flag manifolds one can easily give a hypothetical description of the quantum equivariant cohomology algebras in terms of generalized Toda lattices, but a proof should involve some additional argument.

Our inductive step will make use of the following construction. Consider the subgroup $G' = U_m \times U_{n-m} \subset U_n = G$ and the G'-space $Y = F_m \times F_{n-m}$. The induced G-space (in the sense of 3.6) is nothing but the flag manifold F_n. Its fibration over $G/G' = Gr(n, m)$ sends a flag in $\mathbb{C}^n$ to its m-dimensional component.

Let V_m denote generating volume function for quantum equivariant cohomology of F_m.

Lemma 2.

$$V_n(z, q, c)|_{q_m=0} = Direct\ image\ [V_m(z', q', c') \cdot V_{n-m}(z'', q'', c'')]$$

where

$z = (z_1, ..., z_n)$ *are coordinates on* $H^2((F_n)_G)$ *with respect to the basis* $u_1, ..., u_n$ *(see 3.2)*, $z' = (z_1, ..., z_m), z'' = (z_{m+1}, ..., z_n)$,

$q = (q', q_m, q'') = (q_1, ..., q_m, ..., q_{n-1})$,

c, c' *and* c'' *are Chern classes of* U_n, U_m *and* U_{n-m} *respectively, and "Direct image" refers to the direct image operation* $\mathbb{Z}[c', c''] = H^*(BG') \to H^*(G) = \mathbb{Z}[c]$ *for the bundle* $BG' \to BG$ *with the fiber* $Gr(n, m)$ *(see 3.3)*.

Proof. It is a straightforward corollary of the product and induction formulas: factorization $(F_n)_{U_n} = (F_m)_{U_m} \times (F_{n-m})_{U_{n-m}}$ identifies the basis $(u_1, ..., u_n)$ in the 2-nd equivariant cohomology of the product with the union $(u'_1, ..., u'_m, u''_1, ..., u''_{n-m})$ of such basises of factors since both are the standard generator sets in the cohomology of $(\mathbb{C}P^\infty)^n$, and $p_m \in H^2(F_n)$ is represented by the 1-st Chern class of the determinant line bundle over $Gr(n, m)$ and therefore the vertical rational curves in $F_n \to Gr(n, m)$ are exactly those with $d_m = 0$.

4.3 Theorem 2 implies Theorem 1

Indeed, according to the *restriction* property of equivariant generating volume functions (applied to the trivial subgroup in U_n), if a relation

$$R(\partial/\partial z, q, c)V(z, q, c) = 0$$

is satisfied, then $R(\partial/\partial z, q, 0)$ annihilates the non-equivariant generating volume function $V(z, q, 0)$ and thus the relation $R(u, q, 0) = 0$ holds in $QH^*(F_n)$. This proves

Lemma 3.
$$QH^*(F_n) = QH^*_{U_n}(F_n)/(c_1, ..., c_n).$$

4.4 Equivariant quantum cohomology of $\mathbb{C}P^1$

Lemma 4.
$$QH^*_G(\mathbb{C}P^1) = \mathbb{Z}[u_1, u_2, q, c_1, c_2]/(u_1 + u_2 = c_1, u_1u_2 + q = c_2)$$

Proof. Quantum equivariant cohomology of the projective line F_2 is isomorphic to the quotient algebra of $\mathbb{Z}[u_1, u_2, q, c_1, c_2]$ by the ideal generated by quantum deformations of the relations $u_1 + u_2 = c_1, u_1u_2 = c_2$ in the classical equivariant cohomology. These deformations can be taken

quasi-homogeneous and since $\deg q = 2$, the only possible deformation should replace the RHS in $u_1 u_2 - c_2 = 0$ with a scalar multiple of q.

In order to determine the scalar it suffices to reduce the relations modulo (c_1, c_2), i. e. to compare, by Lemma 3, with relations in the ordinary, non-equivariant quantum cohomology of $F_2 = \mathbb{C}P^1$. Then $u_2 = -u_1$ represents the 1-st Chern class of the "hyperplane" bundle over $\mathbb{C}P^1$, i. e. simply a point. Since the relation $u_2^2 = q$ holds in the quantum cohomology of $\mathbb{C}P^1$ (see 2.7), the scalar coefficient we are looking for equals 1.

4.5   Step of induction

Denote

$$D_n(u, q, \lambda) = \det(A_{n-1} + \lambda)$$

the characteristic polynomial of the $n \times n$-matrix with $u_1, ..., u_n$ on the diagonal, $q_1, ..., q_{n-1}$ right above and $-1, ..., -1$ right under the diagonal.

Lemma 5. *Suppose that the relation*

$$D_k(u, q, \lambda) = \lambda^k + c_1 \lambda^{n-1} + ... + c_k$$

is satisfied identically in λ in the equivariant quantum cohomology of flag manifolds F_k for all $k < n$. Then the relation with $k = n$ is also satisfied modulo q_m for every $m = 1, ..., n-1$.

Proof. First of all, notice that $D_n|_{q_m=0} = D_m(u', q', \lambda) D_{n-m}(u'', q'', \lambda)$ where $(u', u'') = u, (q', 0, q'') = q$.

Denote

$$\Sigma_n = \lambda^n + c_1 \lambda^{n-1} + ... + c_n = (x_1 + \lambda)...(x_n + \lambda)$$

the RHS of the above relation with the Chern classes $c_1, ..., c_n$ written for convenience as elementary symmetric functions of the formal variables $x_1, ..., x_n$. The conclusion of Lemma 5 means that

$$[D_n(\partial/\partial z, q, \lambda) - \Sigma_n(x, \lambda)] \, V_n(z, q, \sigma(x)) \,|_{q_m=0} = 0.$$

It is the same as

$$[D_m(\frac{\partial}{\partial z'}, q', \lambda) D_{n-m}(\frac{\partial}{\partial z''}, q'', \lambda) - \Sigma_n(x, \lambda)] \, [V_n((z', z''), (q), \sigma(x))|_{q_m=0}] = 0.$$

By Lemma 2, the function $V_n|_{q_m=0}$ in the last formula can be replaced with the *Direct image* of

$$V_m(z', q', \sigma(x')) \cdot V_{n-m}(z'', q'', \sigma(x'')),$$

explicitly described in 3.3.

Since the derivations in $D_m D_{n-m}$ are with respect to z', z'' which are not involved into permutations in the operation *Alt*, and the variables x', x'' which are involved do not show up in coefficients of the operators D_m, D_{n-m}, the *Direct image* operation commutes with our differential operator.

Applying the inductive assumption we find that the conclusion of the proposition is equivalent to the identity

$$\Sigma_n(x, \lambda) \text{Direct image } [V_m(x') V_{n-m}(x'')] =$$

$$= \text{Direct image } [(\Sigma_m(x', \lambda)V_m(x'))(\Sigma_{n-m}(x'', \lambda)V_{n-m}(x''))].$$

But

$$\Sigma_m(x', \lambda)\Sigma_{n-m}(x'', \lambda) = (x_1 + \lambda)...(x_n + \lambda) = \Sigma_n(x, \lambda)$$

is totally symmetric in $(x_1, ..., x_n)$!.

Since multiplication by a symmetric function commutes with the alternation operation, we conclude that the required identity does hold.

Combining Lemma 5 with Lemma 1 completes the proof of Theorem 2 from Introduction.

4.6   Volume functions

We have found the relations in quantum cohomology of flag manifolds using general properties of generating volume function. Now we compute the quantum volume functions using our knowledge of the relations and of the classical volume functions.

Proposition. *The quantum equivariant generating volume function $V_n(z, q, c)$ of the flag manifold F_n equals*

$$W_n = \frac{1}{(2\pi i)^n} \int \frac{\exp(z, u)du_1 \wedge ... \wedge du_n}{(\Sigma_1(u, q) - c_1)...(\Sigma_n(u, q) - c_n)}$$

where $\Sigma_i(u, q)$ are the quantum deformations of elementary symmetric functions i. e. the coefficients of the polynomial $\det(A_{n-1} + \lambda)$.

Proof. By the *deformation* property and Proposition in 3.2, the formula holds for $q = 0$. We will prove the formula using the homogeneity property $\deg V_n = -\dim F_n$ (where $\deg z_i = -1, \deg q_i = 2, \deg c_i = i$) and the differential equations

$$\Sigma_i(\partial/\partial z, q)V_n(z, q, c) = c_i V_n(z, q, c), \ i = 1, ..., n.$$

First of all, the function W_n does satisfy the homogeneity condition and the differential equations (due to the famous property of residues).

Due to another property of residues (see [GH]) W_n is an analytic function of its variables and can be expanded into a power series (V_n is a formal series by definition). Represent the difference $V_n - W_n$ as a sum $\sum_{d \geq 0, l \geq 0} R_{d,l}(z)q^d c^l$. The coefficient $R_{d,l}$ is a homogeneous polynomial in z of degree (in the usual sense) $\dim F_n + \sum 2d_i + \sum jl_j$ and $R_{0,0} = 0$ since V_n coincides with W_n at $q = 0$.

Let us pick R as the coefficient of minimal degree among non-zero $R_{d,l}$. The differential equations for $V - W$ mean that

$$\sigma_i(\partial/\partial z)R(z) = \text{some operators applied to } R_{d,l} \text{ with smaller } d, l$$

and hence that $\sigma_i(\partial/\partial z)R(z) = 0, i = 1, ..., n$, since all those $R_{d,l}$ are zeroes. Now the following lemma completes the proof.

Lemma 6. *If all symmetric differential polynomials $S(\partial/\partial z)$ in n variables annihilate a polynomial $R(z)$, then $\deg R \leq \dim F_n$.*

Proof. The quotient of the algebra of all differential polynomials $S(\partial/\partial z)$ by the ideal generated by elementary symmetric functions is canonically isomorphic to the cohomology algebra $H^*(F_n)$. This implies that the ideal containes the power $\mathfrak{m}^{\dim \mathfrak{F}_n + 1}$ of the maximal ideal $\mathfrak{m} = (\partial/\partial \mathfrak{z}_1, ..., \partial/\partial \mathfrak{z}_n)$. This means that all derivatives of R of order $> \dim F_n$ vanish and thus $\deg R \leq \dim F_n$.

Proposition also implies Theorem 3 from Introduction (describing quantum intersection indices), since by definition of V_n

$$\langle f|g \rangle = [f(\partial/\partial z)g(\partial/\partial z)V_n(z, q, c)]|_{z=0}.$$

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