SOLUTIONS

Math 104. Midterm Exam. 02.28.06

Problem 1. Formulate the definition of a Cauchy sequence (full sentences, please).

A sequence (x_n) (of real numbers) is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists a natural number N such that for all $m, n \geq N$ we have $|x_n - x_m| < \epsilon$.

Problem 2. Prove that the product of two convergent sequences converges to the product of their limits.

From the definition of a convergent sequence, it follows immediately that a sequence x_n converges to x if and only if the sequence $\Delta x_n := x_n - x$ converges to 0.

It also follows, that a convergent sequence is bounded.

Next, if a sequence a_n converges to 0, and a sequence b_n is bounded by M, i.e. $|b_n| < M$ for all n, then the product $a_n b_n$ converges to 0. Indeed, for every positive ϵ, find N such that for all $n \geq N$ we would have $|a_n| < \epsilon/M$ (such an N exists because $\lim a_n = 0$). Then for all $n \geq N$ we will have $|a_n b_n| < \epsilon$.

Now, let $\lim s_n = s$, $\lim t_n = t$. Then $s_n = s + \Delta s_n$ and $t_n = t + \Delta t_n$, where $\lim \Delta s_n = 0$ and $\lim \Delta t_n = 0$. Therefore

$$s_n t_n = (s + \Delta s_n)(t + \Delta t_n) = st + t\Delta s_n + s\Delta t_n + (\Delta s_n)(\Delta t_n).$$

Each of the second, third and fourth summand converges to 0 as the product of a bounded (may be constant) sequence and a sequence converging to 0. Therefore, by the properties of sums of convergent sequences, the product $s_n t_n$ converges to st.

Problem 3. Let $[a_n, b_n]$ be a *nested* sequence of segments of the real number axis, i.e. $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$ for each $n = 1, 2, 3, ...$ Prove that the intersection of the segments is non-empty:

$$\cap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset.$$

By the hypothesis, for every $n \leq m$, we have: $a_n \leq a_m \leq b_m \leq b_n$. Therefore (a_n) is a non-decreasing sequence bounded above by each b_m, and (b_m) is a non-increasing sequence bounded below by each a_n. By the property of bounded monotone sequences to converge, $\lim a_n$ and $\lim b_m$ exists and for each k satisfy:

$$a_k \leq \lim a_n \leq \lim b_m \leq b_k.$$

Therefore the segment $[\lim a_n, \lim b_m]$ is non-empty and is contained in every segment $[a_k, b_k]$.Thus, the intersection of all $[a_k, b_k]$ is non-empty.

Problem 4. Does there exist a bounded sequence (s_n) such that

$$\inf\{s_n\} < \liminf s_n < \limsup s_n < \sup\{s_n\}?$$

Yes, for instance, $s_n = (-1)^n(1 + 1/n)$ has the infimum equal to $s_1 = -2$, the supremum equal to $s_2 = 3/2$, and $\limsup = 1$, and $\liminf = -1$.

Problem 5. Prove that if $\sum a_n$ is a convergent series of non-negative numbers, and $p > 1$, then $\sum a_n^p$ converges.

A number a such that $0 \leq a < 1$ satisfies $0 \leq a^p < a$ for all $p > 1$. If the series $\sum a_n$ of non-negative terms converges, then $\lim a_n = 0$. Therefore there exists N such that for all $n \geq N$ we would have $0 \leq a_n < 1$, and hence $\leq a_n^p < a_n$. Since convergence of $\sum a_n$ implies convergence of $\sum a_{N+n}$ (here N is fixed, and $n = 1, 2, 3, ...$, we conclude (by the comparison test), that the series $\sum a_{N+n}^p$ converges, and therefore $\sum a_n^p$ converges as well.