YOUR NAME: A. Givental

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1. Represent bilinear form $B(\mathbf{x}, \mathbf{y}) = x_1y_2 + x_2y_3 + x_3y_1$ as the sum of symmetric and anti-symmetric ones.

`Solution.`

$$x_1y_2 + x_2y_3 + x_3y_1 = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 + x_2y_1 + x_3y_2 + x_1y_3) \\ + \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3).$$

2. Let P_{ij}, $1 \le i < j \le 4$, denote the 2×2-minor of a 2×4-matrix formed by the columns i and j. Prove that $P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0$.

`Solution.` Let (a_1, a_2, a_3, a_4) abd (b_1, b_2, b_3, b_4) be the rows of the 2×4-matrix. Then the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix} = 0,$$

as having identical rows. On the other hand, using Lagrange's Theorem (namely, the cofactor expansion with respect to the first two rows), we find that this determinant is equal to twice $P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23}$.

3. Compute the following determinant:

$$\begin{vmatrix} a & 0 & 0 & 0 & 0 & b \\ 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 \\ 0 & c & 0 & 0 & d & 0 \\ c & 0 & 0 & 0 & 0 & d \end{vmatrix}.$$

`Solution.` Partition the matrix into 4 blocks $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ of size 3×3. Suppose that $d \neq 0$. Then D is invertible, and we can use the First Cool

Formula. We have: $A = aI_3$, $BD^{-1}C = bd^{-1}cI_3$, and hence:

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A - BD^{-1}C)\ \det D = \det\left[(a - bd^{-1}c)I_3\right]\ \det(dI_3).$$

Thus the answer is $(ad - bc)^3$. When $d = 0$, this is also correct. Namely, the answer reduces to $-b^3c^3$, while the matrix becomes triangular with respect to the "southwest–northeast" diagonal. Its determinant is equal to the product of the diagonal entries (i.e. b^3c^3) and the sign of the permutation $\binom{123456}{654321}$. The latter has length 15 and is odd.

4. On the space $\mathbb{R}^4$, consider the bilinear form

$$B(\mathbf{x}, \mathbf{y}) := \begin{vmatrix} 1 & 1 & x_1 & y_1 \\ 1 & 2 & x_2 & y_2 \\ 1 & 3 & x_3 & y_3 \\ 1 & 4 & x_4 & y_4 \end{vmatrix}.$$

Is this form symmetrix? anti-symmetric? neither? Compute the coefficient matrix B of this form.

`Solution.` The determinant changes the sign when the last two columns are transposed, and therefore the bilinear form is anti-symmetric. Using Lagrange's theorem (namely, the cofactro expansion with respect to the first two columns), we find that the 2×2-minors formed by the first two columns are the coefficients. More precisely, there are 6 such minors:

$$M_{12} = 1, M_{13} = 2, M_{14} = 3, M_{23} = 1, M_{24} = 2, M_{34} = 1.$$

The determinant is equal to

$$M_{12} \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} - M_{13} \begin{vmatrix} x_2 & y_2 \\ x_4 & y_4 \end{vmatrix} + \cdots - M_{34} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

Respectively, the bilinear form $B(\mathbf{x}, \mathbf{y}) = \mathbf{x}^t B \mathbf{y}$ has the coefficient matrix

$$B = \begin{bmatrix} 0 & M_{34} & -M_{24} & M_{23} \\ -M_{34} & 0 & M_{14} & -M_{13} \\ M_{24} & -M_{14} & 0 & M_{12} \\ -M_{23} & M_{13} & -M_{12} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 & 1 \\ -1 & 0 & 3 & -2 \\ 2 & -3 & 0 & 1 \\ -1 & 2 & -1 & 0 \end{bmatrix}.$$

5. Prove that for a linear map $A : \mathcal{V} \to \mathcal{W}$, $\operatorname{Ker} A = (\operatorname{Graph} A) \cap \mathcal{V}$.

`Solution.` By definition, the kernel of A consists of all those vectors $\mathbf{x} \in \mathcal{V}$ for which $A\mathbf{x} = \mathbf{0}$. By definition, the the graph of A is a subspace in the direct sum $\mathcal{V} \oplus \mathcal{W}$ consisting of all those pairs $(\mathbf{x}, \mathbf{y}), \mathbf{x} \in \mathcal{V}, \mathbf{y} \in \mathcal{W}$ for which $\mathbf{y} = A\mathbf{x}$. The space $\mathcal{V}$ is identified with the subspace (which we also denote by $\mathcal{V}$) in $\mathcal{V} \oplus \mathcal{W}$ which consists of all pairs $(\mathbf{x}, 0), \mathbf{x} \in \mathcal{V}, \mathbf{0} \in \mathcal{W}$. Its intersection with the graph consists therefore of all those pairs $(\mathbf{x}, \mathbf{0})$ for which $\mathbf{0} = A\mathbf{x}$. Thus this intersection coincides with $\operatorname{Ker} A \subset \mathcal{V} \subset \mathcal{V} \oplus \mathcal{W}$.

`Remark.` This is the linear algebra version of the fact in elementary algebra that the graph of a function $y = f(x)$ intersects the x-axis exactly at the zeroes of the function f.