YOUR NAME: Alexander Givental

Math 110. Midterm I. Feb. 9, 2009

1. Formulate and prove the "toy version" of the Orthogonal Diagonalization Theorem (about quadratic forms on the plane).

`Solution.` See page 11 of the text.

2. A regular triangle ABC is inscribed into a circle of radius R. Prove that for every point X of this circle, $XA^2 + XB^2 + XC^2 = 6R^2$.

`Solution.` Let O denote the center of the circle, Then $\vec{XA} = \vec{OA} - \vec{OX}$, and hence

$$XA^2 = \langle \vec{OA} - \vec{OX}, \vec{OA} - \vec{OX} \rangle = OA^2 + OX^2 - 2\langle \vec{OA}, \vec{OX} \rangle,$$

and similarly for XB and XC. Therefore

$$XA^2 + XB^2 + XC^2 = 3OX^2 + OA^2 + OB^2 + OC^2 - 2\langle \vec{OA} + \vec{OB} + \vec{OC}, \vec{OX} \rangle.$$

Since the triangle ABC is equilateral, we have $\vec{OA} + \vec{OB} = -\vec{OC}$. Since $OA = OB = OC = OX = R$, the result follows.

3. Find all roots of the polynomial $z^4 + 4z^2 + 4$.

`Solution.` $z^4 + 4z^2 + 4 = (z^2 + 2)^2$. Thus there are two roots, $z = \sqrt{-2} = \pm i\sqrt{2}$, of multiplicity 2 each.

4. Find the place of the curve $x^2 + 2xy + y^2 = 2x$ in classification of quadratic plane curves (i.e. find out if it is an ellipse, hyperbola, parabola, two crossing lines, two parallel lines, one line, one point, or the empty set).

`Solution.` We have: $x^2 + 2xy + y^2 = (x + y)^2$, and therefore in new coordinates $X = x + y, Y = 2x$, the curve is given by the equation $X^2 = Y$. Thus it is a parabola.

5. Classify quadratic forms $Q = ax^2$ in one variable with complex coefficients (i.e. if $a \in \mathbb{C}$) up to complex linear changes $x = cy$, $c \in \mathbb{C}$, $c \neq 0$.

`Solution.` When $a = 0$, the quadratic form Q is equal to zero identically, and remais zero under any change of the variable. When $a \neq 0$, take $c = a^{-1/2}$ to transform Q to the form $Q = y^2$. Thus, there are two equivalence classes, whose representatives are 0 and y^2.