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Math 53. Midterm I. September 22, 2006.

Theoretical question *(15 pts)*: Areas and 2×2-determinants.

Given two vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in the plane, the determinant $a_1 b_2 - a_2 b_1$ of the matrix formed by the coordinates of these vectors is equal to the signed area of the parallelogram P formed by these two vectors:

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \pm Area(P).$$

The sign here is "+" is the direction of rotation from a to b is counter-clockwise, and "−" is it is clockwise.

Indeed, let ϕ and ψ be the angles that the vectors a and b respectively make with the positive direction of the 1st coordinate axis, and $\theta = \psi - \phi$ be the angle of rotation from a to b (which is positive, when the direction of this rotation is counter-cklockwise). Then

$$a_1 = |a| \cos \phi, \ a_2 = |a| \sin \phi, \ b_1 = |b| \cos \psi, \ b_2 = |b| \sin \psi,$$

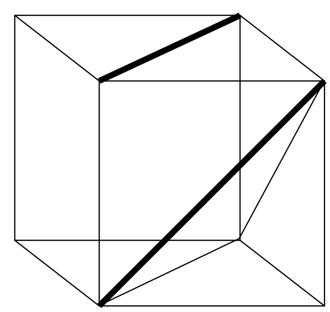
and hence

$$a_1 b_2 - a_2 b_1 = |a| \ |b| \ (\cos \phi \sin \psi - \sin \phi \cos \psi) = |a| \ |b| \ \sin(\psi - \phi) = |a| \ |b| \ \sin \theta.$$

The latter expression coincides with the signed area of the parallelogram P, since one can think of $|a|$ as the base of P, and $|b| \ |\sin \theta|$ as its altitude.

Problem 1. (5 pts)

Find the angle between two
non−intersecting diagonals
in two adjacent faces of a cube.



Solution. Replace the diagonal of the top face of the cube (see the figure) with the diagonal of the bottom face and such that this new diagonal is parallel to the old diagonal. Then the new diagonal makes the same angle as the old one with the diagonal in the front face shown in the figure. We claim that the angle is 60°. Indeed, consider the triangle fromed by these two diagonals and one more, in the right face of the cube. The triangle is equilateral since all sides of it are diagonals of equal squares. Therefore each angle of this triangle measures 60°.

Problem 2. ***(10 pts)***
(a) Find an equation of the plane passing through the points $P(1,2,0)$, $Q(0,1,-1)$, and $R(1,1,1)$.
(b) Compute the area of the triangle PQR.

Solution. (a) The cross-product of the vectors $\vec{QP} = (1,1,1)$ and $\vec{QR} = (1,0,2)$ is equal to $\vec{N} = (2,-1,-1)$. It is perpendicular to the plane passing through P, Q, R, and therefore the equation of the plain has the form $2x - y - z = c$ where c can has to be chosen ss that one of the points (say R) satisfies the equation. We have $c = 2 - 1 - 1 = 0$, i.e. the equation of the plane in question is $2x = y + z$.

(b) The area of the triangle PQR is equal to a half of the area of the parallelogram formed by $\vec{QP}$ and $\vec{QR}$, which in its turn is equal to the length of the cross-product $\vec{N}$. Thus the area in question is equal to

$$\frac{1}{2}|\vec{N}| = \frac{1}{2}\sqrt{4+1+1} = \frac{\sqrt{6}}{2} = \sqrt{\frac{3}{2}}.$$

Problem 3. ***(5 pts)*** Graph the plane curve $x^2 + 2x = 4y^2 - 4y$.

Solution. Completing squares, we find:

$$x^2 + 2x - 4y^2 + 4y = (x+1)^2 - 1 - (2y-1)^2 + 1 = (x+1)^2 - (2y-1)^2 = (x+2y)(x-2y+2).$$

Therefore the curve in question is the union of two lines: $x = -2y$, and $x = 2y - 2$, intersecting at the point $(x,y) = (-1, 1/2)$.

Problem 4. ***(5 pts)***
Identify the surface given in spherical coordinates by the equation $\rho^2 \cos 2\phi = 0$.
Solution. The surface is a cone of revolution about the z-axis, since

$$\rho^2 \cos 2\phi = \rho^2 \cos^2\phi - \rho^2 \sin^2\phi = z^2 - r^2 = z^2 - x^2 - y^2.$$