Math 140. Midterm exam. 03.03.05. Solutions.

Problem 1. Let C be the plane curve $(x(t), y(t) = (t^2, t^5)$, and let $r(t) = \Phi(x(t), y(t))$ be another plane curve obtained from C by a possibly non-linear smooth map Φ from the plane to itself. Denote k the limit at $t \to 0$ of curvature of the curve $t \mapsto r(t)$. Find the range of possible values of k when Φ varies.

The curve is the graph of the double-valued function $y = \pm x^{5/2}$ which has the 2-nd derivative well-defined and continuous including the point $x = 0$ and equal to 0 at this point (since $5/2 > 2$). This implies that $k = 0$ (we've found this in one of the homeworks) and that the line $y = 0$ (call it L) is the "circle" of infinite radius best approximating the curve C near the origin. Application of a differentiable change of variables Φ transforms C and L to new curves C' and L' which still approximate each other up to order 2 near the image $\Phi(0, 0)$ of the origin and thus have the same curvature at this point. Since the curvature of L' can be any number (e.g. $\Phi(x, y) = (x, y + ax^2/2)$ transforms $y = 0$ to the parabola $y = ax^2/2$ which has the curvature a at the origin), the same applies to C'.

Problem 2. Find all those real numbers which can be the values of the total curvature of closed regular space curves of non-constant curvature.

Any closed regular space curve has total curvature $\geq 2\pi$ by Fenchel's theorem. Any closed regular convex *plane* curve of index 1 other than a circle has non-constant curvature and has total curvature equal to 2π. Thus the value 2π is possible. Any value of the total curvature greater than 2π is also possible and easy to achieve (in a thousand different ways) by locally modifying a curve C with the total curvature equal to 2π. For instance, replace a small arc on C by the same arc placed on a parallel plane and connected with the remaining curve by two helices with appropriate parameters. Notice that the velocity curves of helices are arbitrary non-equatorial circles on the unit sphere, and the total curvatures — the lengths of these circles. So varying the helices with n twists, one can increase the total curvature by any number between 0 and $4\pi n$ (twice $2\pi n$), where $n = 1, 2, \ldots$ is arbitrary.

Problem 3. Compute the first fundamental form of the parameterized surface $x = u + v,\ y = u + 2v,\ z = u + 3v$ and find the surface area of the region $u^2 + v^2 \leq r^2$.

The Riemannian metric is $(dx)^2 + (dy)^2 + (dz)^2 =$

$$(du + dv)^2 + (du + 2dv)^2 + (du + 3dv)^2 = 3(du)^2 + 2 \cdot 6(du)(dv) + 14(dv)^2.$$

The area is

$$\int\int_{u^2+v^2\leq r^2} \sqrt{3 \cdot 14 - 6^2}\, dudv = \sqrt{6} \int\int_{u^2+v^2\leq r^2} dudv = \pi r^2 \sqrt{6},$$

i.e. $\sqrt{6}$ times the Euclidean area of the disc of radius r.

Problem 4. Compute the total geodesic curvature of the closed curve $x = \cos t,\ y = \sin t,\ z = 1,\ 0 \leq t \leq 2\pi$ on the surface of the cone $x^2 + y^2 = z^2$.

The part of the cone bounded by the circle $z = 1$ can be obtained by bending a piece of paper. Developing the paper back to the plane, we get a sector bounded by the circular arc of length 2π and of radius $\sqrt{2}$ (and therefore comprising the central angle of $2\pi/\sqrt{2}$ radian). Thus the geodesic curvature of the circle $z = 1$ (i.e. the ordinary curvature of the arc representing it on the Euclidena plane) is $1/\sqrt{2}$. Since the arc length is 2π, the total geodesic curvature of our curve is equal to $2\pi/\sqrt{2} = \sqrt{2}\pi$.

For those willing to double-check that our development of the cone is an isometry, consider the explicit parameterization of the cone by the sector $0 \leq \arctan v/u \leq 2\pi/\sqrt{2}$, $u^2 + v^2 \leq 2$:

$$x = \sqrt{(u^2 + v^2)/2}\ \cos(\sqrt{2}\tan^{-1}(v/u)), \tag{1}$$

$$y = \sqrt{(u^2 + v^2)/2}\ \sin(\sqrt{2}\tan^{-1}(v/u)), \tag{2}$$

$$z = \sqrt{(u^2 + v^2)/2}, \tag{3}$$

and compute the Riemannian metric:

$$(dx)^2 + (dy)^2 + (dz)^2 = \frac{(udu + vdv)^2}{u^2 + v^2} + \frac{(udv - vdu)^2}{u^2 + v^2} = (du)^2 + (dv)^2.$$