Math 104: Introduction to Analysis

SOLUTIONS

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HOMEWORK 12

31.2 Find the Taylor series for $\sinh x = (e^x - e^{-x})/2$ and $\cosh x = (e^x + e^{-x})/2$.

`Solution.` The result

$$\sinh x = \sum_{n\geq 1} \frac{x^{2n-1}}{(2n-1)!}, \quad \cosh x = \sum_{n\geq 0} \frac{x^{2n}}{(2n)!}$$

follows easily from either the expansion $e^x = \sum x^n/n!$, or from $(\sinh x)' = \cosh x$, $(\cosh x)' = \sinh x$, togehter with $\sinh(0) = 0$, $\cosh(0) = 1$. Convergence of each series for any x follows from the ratio test:

$$x^2/2n(2n+1) \to 0 < 1, \text{ and } x^2/(2n+1)(2n+2) \to 0 < 1 \text{ as } n \to \infty.$$

23.2cd Determine the radius and exact interval of convergence for the series: (c) $\sum x^{n!}$ and (d) $\sum 3^n x^{2n+1}/\sqrt{n}$.

`Solution.` (c) When $|x| \geq 1$, the series diverges since its general term $x^{n!}$ does not tend to 0 as $n \to 0$. For $|x| < 1$ the series converges absolutely by comparison with $\sum |x|^m$ (note that $\sum x^{n!} = \sum a_m x^m$ with $a_m = 1$ when $m = n!$, and $a_m = 0$ when $m \neq n!$). Thus the convergence radius is 1, and the exact interval of convergence is $(-1, 1)$.

(d) The convergence radius is

$$R = 1/\lim(3^n/\sqrt{n})^{1/(2n+1)} = \frac{1}{\sqrt{3}} \lim(3n)^{1/(4n+2)} = \frac{1}{\sqrt{3}}.$$

When $x = \pm 1/\sqrt{3}$, the series turns into $\sum \pm 1/\sqrt{3n}$, which is known to diverge to $\pm\infty$. Thus the interval of convergence is $(-1/\sqrt{3}, 1/\sqrt{3})$.

23.6b. Give an example of a series whose interval of convergence is exactly $(-1, 1]$.

`Solution.` The series $\sum_{n>0}(-x)^n/n$ converges (to $-\ln(1+x)$) when $|x| < 1$ (by the root test), diverges at $x = -1$ (since $\sum 1/n = \infty$, and converges at $x = +1$ as an alternating series whose terms $\pm 1/n$ monotonically tend to 0 in the absolute value.

23.8. Show that $f_n(x) := n^{-1} \sin nx$ are differentiable, tend to 0 for all $x \in \mathbf{R}$, but $\lim f_n'(x)$ need not exist (at $x = \pi$ for instance).

`Solution.` Indeed, $f_n(x)$ tends to 0 since $|n^{-1} \sin nx| \neq 1/n \to 0$ as $n \to \infty$. However the sequence $f_n'(x) = \cos nx$ turns into $(-1)^n$ at $x = \pi$ which has no limit as $n \to \infty$.