Math 104: Introduction to Analysis

SOLUTIONS

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HOMEWORK 1

1.1. *Prove that* $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ *for all* $n \in \mathbf{N}$.

Put $f(n) = n(n+1)(2n+1)/6$. Then $f(1) = 1$, i.e the theorem holds true for $n = 1$. To prove the theorem, it suffices to assume that it holds true for $n = m$ and derive it for $n = m + 1$, $m = 1, 2, 3, ...$

We have

$$f(m+1) - f(m) = \frac{1}{6}(m+1)[(2m+3)(m+2) - m(2m+1)]$$
$$= \frac{1}{6}(m+1)(6m+6) = (m+1)^2.$$

By the induction hypothesis, $f(m) = \sum_{k=1}^{m} k^2$, and therefore

$$f(m+1) = f(m) + (m+1)^2 = \sum_{k=1}^{m+1} k^2.$$

1.9 *Decide for which n the inequality* $2^n > n^2$ *holds true, and prove it by mathematical induction.*

The inequality is false $n = 2, 3, 4$, and holds true for all other $n \in \mathbf{N}$. Namely, it is true by inspection for $n = 1$, and the equality $2^4 = 4^2$ holds true for $n = 4$. Thus, to prove the inequality for all $n \geq 5$, it suffices to prove the following inductive step:
For any $n \geq 4$, if $2^n \geq n^2$, then $2^{n+1} > (n+1)^2$.

This is not hard to see: $2^{n+1} = 2 \cdot 2^n \geq 2n^2$, which is greater than $(n+1)^2$ provided that $(n+1) < \sqrt{2}n$ i.e. when $n > 1/(\sqrt{2}-1) = \sqrt{2}+1$, which includes all integers $n \geq 4$.

1.12bc. *Put* $\binom{n}{k} := n!/k!(n-k)!$, *prove (a)* $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ *for* $k = 1, ..., n$, *and (b) derive the binomial theorem by induction.*

(b) Note that

$$\frac{1}{k} + \frac{1}{n-k+1} = \frac{(n-k+1)+k}{k(n-k+1)} = \frac{n+1}{k(n-k+1)}.$$

Therefore

$$\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!}{(k-1)!(n-k)!}\left[\frac{1}{k} + \frac{1}{n-k+1}\right]$$
$$= \frac{n!}{(k-1)!(n-k)!}\left[\frac{n+1}{k(n-k+1)}\right] = \frac{(n+1)!}{k!(n-k+1)!}.$$

(c) For $n = 1$ we have $(a+b)^n = a + b = \binom{1}{1}a + \binom{1}{1}b$.

Suppose for some $n \geq 1$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Then

$$(a+b)^{n+1} = (a+b)\sum_{l=0}^{n} \binom{n}{l} a^l b^{n-l} = \sum_{k=0}^{n+1} \left[\binom{n}{k-1} + \binom{n}{k}\right] a^k b^{n+1-k}$$
$$= \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}.$$

2.5. *Show that* $[3+\sqrt{2}]^{2/3}$ *does not represent a rational number.*

Suppose it does represent a rational number q. Then $q^3 = [3+\sqrt{2}]^2 = 9+6\sqrt{2}+2 = 11+6\sqrt{2}$. Then $\sqrt{2} = (q^3-11)/6 \in \mathbf{Q}$, which contradicts irrationality of $\sqrt{2}$.

HOMEWORK 2

3.6b. *Use induction to prove*

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

for n numbers $a_1, a_2, \ldots, a_n$.

1° For $n = 1$, we need $|a_1| \leq |a_1|$, which is true tautologically.

2° Suppose the required inequality is true for $n = k$. Then for $n = k+1$, using the triangle inequality, we obtain:

$$|a_1+\cdots+a_k+a_{k+1}| \leq |a_1+\cdots+a_k| + |a_{k+1}| \leq |a_1|+\cdots+|a_k|+|a_{k+1}|.$$

Thus the required inequality holds true for any $n = 1, 2, \ldots$

4.4. *Find infima of sets:*

(a) $\inf[0,1] = 0$
(b) $\inf(0,1) = 0$
(c) $\inf[2,7] = 2$
(d) $\inf\{\pi, e\} = e$
(e) $\inf\{\frac{1}{n} :\ n \in \mathbf{N}\} = 0$
(f) $\inf\{0\} = 0$
(g) $\inf[0,1] \cup [2,3] = 0$
(h) $\inf \cup_{n=1}^{\infty} = [2n, 2n+1] = 2$
(i) $\inf \cap_{n=1}^{\infty} [-\frac{1}{n}, 1+\frac{1}{n}] = 0$
(j) $\inf\{1 - \frac{1}{3^n} :\ n \in \mathbf{N}\} = \frac{2}{3}$
(k) $\inf\{n + \frac{(-1)^n}{n} :\ n \in \mathbf{N}\} = 0$
(l) $\inf\{r \in \mathbf{Q} :\ r < 2\} = -\infty$
(m) $\inf\{r \in \mathbf{Q} :\ r^2 < 4\} = -2$
(n) $\inf\{r \in \mathbf{Q} : r^2 < 2\} = -\sqrt{2}$
(o) $\inf\{x \in \mathbf{R} :\ x < 0\} = -\infty$
(p) $\inf\{1, \frac{\pi}{3}, \pi^2, 10\} = 1$
(q) $\inf\{0, 1, 2, 4, 8, 16\} = 0$
(r) $\inf \cap_{n=1}^{\infty} (1 - \frac{1}{n}, 1 + \frac{1}{n}) = 1$
(s) $\inf\{\frac{1}{n} :\ n \in \mathbf{N}$ is prime$\} = 0$
(t) $\inf\{x \in \mathbf{R} :\ x^3 < 8\} = -\infty$
(u) $\inf\{x^2 : x \in \mathbf{R}\} = 0$
(v) $\inf\{\cos \frac{\pi n}{3} :\ n \in \mathbf{N}\} = -1$
(w) $\inf\{\sin \frac{\pi n}{3} :\ n \in \mathbf{N}\} = -\frac{\sqrt{3}}{2}$

4.12. *Prove that given $a < b$, there exists an irrational x such that $a < x < b$.* `Hint:` *first show that $r + \sqrt{2}$ is irrational when $r \in \mathbf{Q}$.*

Following the hint, we prove *by contradiction (reductio ad absurdum)* that $r + \sqrt{2}$ is irrational when $r \in \mathbf{Q}$. Indeed, if for a rational r, the number $x = r + \sqrt{2}$ were rational, then $\sqrt{2} = x - r$ would have been rational, which is false.

Now, using density of $\mathbf{Q}$ in $\mathbf{R}$, find a rational r such that $a - \sqrt{2} < r < b - \sqrt{2}$. Then $x = r + \sqrt{2}$ is irrational, and such that $a < x < b$.

4.14b. *For bounded subsets $A, B \subset \mathbf{R}$, and $S = \{a + b : \ a \in A, \ b \in B\}$, prove that* $\inf \ S = \inf \ A + \inf \ B$.

For any $a \in A$ and $b \in B$, we have:

$$a \geq \inf \ A, \ b \geq \inf \ B, \text{ and hence } a + b \geq \inf \ A + \inf \ B.$$

Therefore $x := \inf \ A + \inf \ B$ is a lower bound of S.

To prove that x is the greatest lower bound, let us show that for any $\epsilon > 0$ we can find $s \in S$ such that $x \leq s < \epsilon$ (which would guarantee that no lower bound of S greater than x exists). For this, find $a \in A$ and $b \in B$ such that $\inf \ A \leq a < \epsilon/2$ and $\inf \ B \leq b < \epsilon/2$. Then $s = a + b \in S$ will satisfy $x \leq s < e$ indeed.

4.15. *Let $a, b \in \mathbf{R}$. Show that if $a \leq b + \frac{1}{n}$ for all $n \in \mathbf{N}$, then $a \leq b$.*

Let us argue by *reductio ad absurdum*. Suppose that $a > b$. Then $a - b > 0$, and therefore, by the Archimedian property of $\mathbf{R}$, there exists $n \in \mathbf{N}$ such that $a - b > \frac{1}{n}$. For this n, we have: $a > b + \frac{1}{n}$, which contradicts the hypotheses.

HOMEWORK 3

8.7. *Show that $s_n = \cos(n\pi/3)$ does not converge.*

For $n = 1, ..., 6$ the terms of the sequence are $1/2$, $-1/2$, -1, $-1/2$, $1/2$, 1, which then repeat periodically. Thus for any number s, and any N one can find $n > N$ such that $s_n = 1$, hence $s_{n+3} = -1$, an therefore, by the triangle inequality, either $|s_n - s| \geq 1$, or $|s_{n+3} - s| \geq 1$.

8.8.c *Prove that* $\lim[\sqrt{4n^2 + n} - 2n] = 1/4$.

$$\sqrt{4n^2 + n} - 2n = \frac{n}{\sqrt{4n^2 + n} + 2n} = \frac{1}{2\sqrt{1 + \frac{1}{4n}} + 2}.$$

For any $1 < a$, we have $1 < a < a^2$, and therefore

$$1 \leq \lim \sqrt{1 + \frac{1}{2n}} \leq \lim \left[1 + \frac{1}{2n}\right] = 1 + 0 = 1.$$

Applying other theorems about behavior of limits under arithmetic operations with sequences, we conclude that

$$\lim \frac{1}{2\sqrt{1+\frac{1}{4n}}+2} = \frac{1}{2\cdot 1+2} = \frac{1}{4}.$$

9.5. *Let $t_1 = 1$ and $t_{n+1} = (t_n^2+2)/2t_n$ for $n \geq 1$. Assume that t_n converges and find the limit.*

Suppose that $t := \lim t_n$ exists. Then $\lim t_{n+1} = t$ as well. For all n, we have: $2t_n t_{n+1} = t_n^2 + 2$. Passing to the limit and using theorems about limits of sums and products of sequences, we conclude that $2t^2 = t^2 + 2$. (In other words, the limit t if exists, must be a fixed point of the function $t_{n+1} = (t_n^2+2)/2t_n$, namely: $t = (t^2+2)/2t$.) We find therefore $t = \pm\sqrt{2}$. Since the sequence t_n with the initial value $t_1 = 1$ stays positive for all n, the limit has to be $+\sqrt{2}$.

`Remark.` Trying this method of computing $\sqrt{2}$, we find: $t_1 = 1$, $t_2 = 3/2$, $t_3 = 17/12$, which is already a good approximation, since $(17/12)^2 = 289/144 = 2\frac{1}{144}$.

9.12. *Assume all $s_n \neq 0$ and that the limit $L = \lim |s_{n+1}/s_n|$ exists. Show that if $L < 1$, then $\lim s_n = 0$.*

Pick a such that $L < a < 1$. For $\epsilon = a - L > 0$, there exists N such that for all $n \geq N$ the ratio s_{n+1}/s_n differs from L by no more than ϵ, and hence $|s_{n+1}/s_n| < L + \epsilon = a < 1$. In particular, $|s_{N+1}| < a|s_N|$, $|s_{N+2}| < a|s_{N+1}| < a^2|s_N|$, and so on, i.e. by induction, $|s_{N+n}| < a^n|s_N|$ for all $n \in \mathbf{N}$. We conclude:

$$\lim_{n\to\infty} |s_n| = \lim_{n\to\infty} |s_{N+n}| \leq \lim_{n\to\infty} a^n |s_N| = |s_N| \lim_{n\to\infty} a^n = 0$$

when $|a| < 1$.

9.15. *Show that $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbf{R}$.*

Put $s_n = a^n/n!$ and find that $s_{n+1}/s_n = a/(n+1)$ tends to 0 as $n \to \infty$. Therefore, by the previous exercise, $\lim s_n = 0$. (In other words, $n!$ grows faster than any exponential sequence a^n.)

HOMEWORK 4

10.6. (a) *Let (s_n) be a sequence such that $|s_{n+1} - s_n| < 2^{-n}$ for all $n \in \mathbf{N}$. Prove that (s_n) is a Cauchy sequence and hence a convergent sequence.*

For any $m > n$, we have

$$|s_m - s_n| \leq \sum_{n\leq k<m} |s_{k+1} - s_k| < \sum_{n\leq k<m} 2^{-k} = 2^{-n+1} - 2^{-m} < 2^{-n+1}.$$

Therefore, for any given $\epsilon > 0$, choosing N such that $2^{-N+1} < \epsilon$, we will have $|s_m - s_n| < \epsilon$ for all $m \geq n \geq N$. Thus (s_n) is a Cauchy sequence.

(b) *Is the result (a) true if we only assume that* $|s_{n+1} - s_n| < 1/n$ *for all* $n \in \mathbf{N}$*?*

No. To construct a counter-example, let us prove first that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Indeed, for each k, there are $2^k - 2^{k-1} - 2^k = 2^{k-1}$ numbers of the form $1/n$ between $1/(2^{k-1}+1)$ and $1/2^k$. Each of them is at least as large as $1/2^k$, and hence they and up to $2^{k-1}/2^k = 1$. Thus the sum of the first m such groups is at least $m/2$, i.e.

$$\sum_{n=2}^{2^m} \frac{1}{n} \geq \sum_{k=1}^{m} \frac{1}{2} = \frac{m}{2}.$$

Thus the sum of finitely many many terms of the series becomes greater than any positive integer m when the number of the summands increases.

Now, put $s_n := \sum_{k=1}^{n} \frac{1}{k}$, so that $\lim s_n = \infty$, but

$$|s_{n+1} - s_n| = \frac{1}{n+1} < \frac{1}{n} \text{ for all } n \in \mathbf{N}.$$

10.7. *Let S be a bounded non-empty subset of* $\mathbf{R}$*, and suppose* $\sup S \notin S$*. Prove that there is a non-decreasing sequence* (s_n) *of points in S such that* $\lim s_n = \sup S$.

For each $n \in N$, construct $s_n \in S$ such that $\sup S - s_n < 1/n$ and $s_n > s_{n-1}$ for $n > 1$. Then (s_n) will be an increasing sequence converging to $\sup S$.

Start by picking $s_1 \in S$ such that $\sup S - s_1 < 1$. This is possible since otherwise $\sup S - 1 < \sup S$ is not an upper bound of S. Proceed by induction: suppose that $s_1 < ... < s_{n-1}$ with required properties have been found. Since $\sup S \notin S$, we have $s_{n-1} < \sup S$. Therefore there exists $s_n \in S$ such that

$$\sup S \geq s_n > s_{n-1} \text{ and } \sup S - s_n < 1/n,$$

which is possible since neither $s_{n-1} < \sup S$ nor $\sup S - 1/n < \sup S$ is an upper bound of S.

12.3. *Let (s_n) and (t_n) be the following two sequences that repeat in cycles of four:*

$$(s_n) = (0,1,2,1,0,1,2,1,...), \quad (t_n) = (2,1,1,0,2,1,1,0,...).$$

Then:

$$\begin{aligned}
&(a) \ \liminf s_n + \liminf t_n = 0 + 0 = 0, \\
&(b) \ \liminf (s_n + t_n) = 1, \\
&(c) \ \liminf s_m + \limsup t_n = 0 + 2 = 2, \\
&(d) \ \limsup (s_n + t_n) = 3, \\
&(e) \ \limsup s_n + \limsup t_n = 2 + 2 = 4, \\
&(f) \ \liminf s_n t_n = 0, \\
&(e) \ \limsup s_n t_n = 2.
\end{aligned}$$

Problem. *Let p_n denote the semiperimeter of a regular $3x2^n$- gon (i.e. 3-angle, 6-gon, 12-gon, 24-gon, etc.) inscribed into a circle of radius 1. Prove that the sequence p_n converges, and that the limit (commonly called π) is greater than 3.*

`Solution.` When the number of sides of a regular polygon doubles, each side of the m-gon is replaced by a broken line consisting of two adjacent sides of the $2m$-gon connecting the same endpoints. Since, by the triangle inequality, a broken line is longer than the straight segment connecting the same endpoints, we conclude that the sequence p_n of semiperimeters is increasing. It is easy to see from elementary geometry that the side of a regular hexagon inscribed into a unit circle has length 1, and therefore $p_1 = 3$. Thus, it suffices to show that the sequence p_n is bounded above, to conclude that the limit π exists and is greater than 3. In fact it is not hard to show that the perimeter of any *convex* polygon does not exceed the perimeter of any polygon containing it. To see this, go around the convex polygon clockwise and extend each side of it forward up to its first intersection with the boundary of the enclosing polygon. Then write down the "triangle" inequality estimating above each of the extended sides by the length of the broken line connecting its endpoints and consisting of the extending segment of the previous side, and a part of the perimeter of the enclosing polygon. Summing up all these inequalities, we obtain the required inequality between the perimeters of the enclosed and enclosing polygons (since the extending segments occur on both sides of the inequality and thus cancel out). This implies that each p_n is smaller than the semiperimeter of any polygon enclosing the unit disk, (e.g. the square of size 2×2, whose semiperimeter equals 4). In particular, we conclude that $\pi < 4$.

HOMEWORK 5

Problem. *A set is called closed if it contains all its subsequential limits (see p. 72). A set is called open if its complement is closed. Prove that a set is open if and only if together with any point, it contains some open interval containing this point.*

Suppose that $x \in S$ is not contained in S together with any open interval. Then for any $n \in \mathbf{N}$ there exists $x_n \notin S$ such that $|x_n - x| > 1/n$. The sequence (x_n) is in the complement of S and converges to x, which is not in the complement. Thus the complement of S is not closed.

Vice versa, suppose the complement of S is not closed, i.e. there exists a sequence (x_n) in the complement of S which converges to $x \in S$. Then any open interval containing x will contain some elements of the sequence (x_n) and thus will not lie in S.

11.9b. *Is there a sequence* (s_n) *such that* $(0,1)$ *is its set of subsequential limits?*

No, the set of subsequential limits of any set must be closed, but the interval $(0,1)$ (not including its endpoints) is not closed.

12.10. *Prove that* (s_n) *is bounded if and only if* $\limsup |s_n| < +\infty$.

If $\limsup s_n = +\infty$, then there exists a subsequence (s_{n_k}) such that $\lim |s_{n_k}| = +\infty$, and therefore this subsequence is unbounded. *Vice versa*, if (s_n) is unbounded, then for any $k \in \mathbf{N}$ there exists s_{n_k} such that $|s_{n_k}| > k$. We may assume that $n_1 < n_2 < ... < n_k < ...$, and thus get a subsequence (s_{n_k}) such that $\lim |s_{n_k}| = +\infty$, i.e. $\limsup |s_n| = +\infty$.

14.4b. *determine if the series* $\sum[\sqrt{n+1} - \sqrt{n}]$ *converges.*

Partial sums of the series are

$$\sqrt{2}-\sqrt{1}+\sqrt{3}-\sqrt{2}+...+\sqrt{n}-\sqrt{n-1}+\sqrt{n+1}-\sqrt{n} = \sqrt{n+1}-\sqrt{1},$$

and form a sequence that tends to $+\infty$. Thus the series diverges.

14.10. *Find a series* $\sum a_n$ *which diverges by the Root Test, but for which the Ratio Test gives no information.*

Consider the series $\sum a_n := \sum 2^{(-1)^n n}$. Applying the Root Test we get the sequence $|a_n|^{1/n} = 2^{(-1)^n}$ which consists of two constant subsequences 2 and 1/2, and therefore has $\limsup |a_n|^{1/n} = 2 > 1$. Thus the series diverges. Applying the Root Test we get the sequence $|a_n/a_{n-1}| = 2^{(-1)^n(2n-1)}$ which consists of two subsequences: 2^{2n-1} for even n, and $1/2^{2n-1}$ for odd n, converging respectively to $+\infty$ and to 0. Thus

$$\liminf |a_n/a_{n-1}| = 0 < 1 < +\infty = \limsup |a_n/a_{n-1}|,$$

i.e. the Ratio Test is inconclusive.

HOMEWORK 6

(a)
$$\sum_{n\geq 1} \frac{1}{\sqrt{n(n+1)}}$$
converges by the comparison test: $1/n(n+1) < 1/n^2$ for all $n \in \mathbf{N}$.

(b)
$$\sum_{n\geq 1} \frac{(n!)^2}{(2n)!}$$
converges by the ratio test: $\lim |a_n/a_{n-1}| = \lim n^2/2n(2n-1) = 1/4 < 1$.

(c)
$$\sum_{n\geq 1} \frac{n!}{n^n}$$
converges by the ratio test: $\lim |a_{n+1}/a_n| = \lim(1+1/n)^{-n} = e^{-1} < 1$.

(d)
$$\sum_{n\geq 1} \frac{(n!)^2}{2^{n^2}}$$
converges by the ratio test: $\lim |a_n/a_{n-1}| = \lim n^2/2^{2n-1} = 0 < 1$.

(e)
$$\frac{1000}{1} + \frac{1000 \cdot 1001}{1 \cdot 3} + \frac{1000 \cdot 1001 \cdot 1002}{1 \cdot 3 \cdot 5} + \cdots$$
converges by the ratio test: $\lim |a_n/a_{n-1}| = \lim(1000+n)/(2n+1) = 1/2 < 1$.

(f)
$$\sum_{n\geq 1} (2^{1/2} - 2^{1/3})(2^{1/2} - 2^{1/5}) \cdots (2^{1/2} - 2^{1/(2n+1)})$$
converges by the ratio test: $\lim |a_n/a_{n-1}| = \lim(2^{1/2} - 2^{1/2n+1}) = \sqrt{2} - 1 < 1$.

(g)
$$\sum_{n\geq 1} \frac{n^2}{(2+\frac{1}{n})^n}$$
converges by the root test: $\lim |a_n|^{1/n} = \lim n^{2/n}/(2+1/n) = 1/2 < 1$.

(h)
$$\sum_{n\geq 1} \frac{n^{n+\frac{1}{n}}}{(n+\frac{1}{n})^n}$$
diverges since $\lim a_n = \lim n^{1/n}/(1+1/n^2)^n > 1/e > 0$.

(i)
$$\sum_{n=1}^{\infty} a_n \quad \text{where } a_n := \begin{cases} 1/n & \text{if } n = m^2 \\ 1/n^2 & \text{if } n \neq m^2. \end{cases}$$

Partial sums of this series form an increasing sequence which is bounded by the sum of $\sum_{n=1}^{\infty} 1/n^2$ with $\sum_{m=1}^{\infty} 1/m^2$. Thus the series converges.

(j) $\sqrt{2}+\sqrt{2-\sqrt{2}}+\sqrt{2-\sqrt{2+\sqrt{2}}}+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}+\cdots$

converges by the ratio test. Indeed, $a_n = \sqrt{2-b_n}$, where $b_0 = 0$, $b_n = \sqrt{2+b_{n-1}} > 0$ for $n > 0$. Then

$$a_n = \frac{\sqrt{2-b_n}\sqrt{2+b_n}}{\sqrt{2+b_n}} = \frac{\sqrt{4-b_n^2}}{\sqrt{2+b_n}} = \frac{\sqrt{2-b_{n-1}}}{\sqrt{2+b_n}} = \frac{a_{n-1}}{\sqrt{2+b_n}}.$$

Thus $\limsup |a_n/a_{n-1}| \le 1/\sqrt{2} < 1$.

HOMEWORK 7

17.7b. Prove that $|x|$ is a continuous function on $\mathbf{R}$.

`Solution.` $|x|$ is continuous at any $x \neq 0$ since it coincides with x for $x > 0$ and with $-x$ for $x < 0$. At $x = 0$, the function $f(x) = |x|$ is continuous because for any $\epsilon > 0$ we have: $|x - 0| < \epsilon$ implies $|f(x) - f(0)| = |x| < \epsilon$.

17.8c. Prove that if f and g are continuous at x_0, then $\min(f, g)$ is continuous at x_0.

`Solution.` According to 17.8a, $\min(f, g) = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$, and therefore continuity of $\min(f, g)$ follows from the property of the absolute value function (17.7b) and the sum, difference, scalar multiple, and composition of continuous functions to be continuous.

17.12b. Let f and g be continuous real-valued functions on (a, b) such that $f(r) = g(r)$ for each rational number r in (a, b). Prove that $f(x) = g(x)$ for all $x \in (a, b)$.

`Solution.` For any $x \in (a, b)$ there exists a sequence of rational numbers r_n such that $\lim f_n = x$. Then $f(x) = \lim f(r_n) = \lim g(r_n) = g(x)$.

17.13b. Let $h(x) = x$ for all $x \in \mathbf{Q}$ and $h(x) = 0$ for all $x \in \mathbf{R} - \mathbf{Q}$. Show that h is continuous at $x = 0$ and no other point.

`Solution.` For any $\epsilon > 0$, if $|x - 0| < \epsilon$, then $|h(x) - h(0)|$ is either 0 (if x is irrational) or $|x|$ (if x is rational, and thus in both cases is smaller than ϵ. This proves continuity of h at $x = 0$. For any other x, consider two sequences with the limit x, one (r_n) consisting of rational numbers, and another x_n consisting of irrational numbers. Then $\lim h(x_n) = 0$, and $\lim h(r_n) = x \neq 0$. This proves discontinuity of h at $x \neq 0$.

17.14. For each rational number x write $x = p/q$ where p, q are integers with no common factors and $q > 0$, and then define $f(x) = 1/q$. Also define $f(x) = 0$ for all $x \in \mathbf{R} - \mathbf{Q}$. Show that f is continuous at each point of $\mathbf{R} - \mathbf{Q}$ and discontinuous at each point of $\mathbf{Q}$.

`Solution.` For a rational $x = p/q$ (as above), find a sequence x_n of irrational numbers such that $\lim x_n = x$. Then $\lim f(x_n) = 0$, but $f(x) = 1/q \neq 0$, i.e. f is discontinuous at x. For an irrational x, and any $\epsilon > 0$, let $\delta > 0$ be the distance from x to the closest irreducible fraction p/q with the denominator $q \leq 1/\epsilon$ (there are at most finitely many such fractions on any bounded interval). Then for any x' such that $|x' - x| < \delta$ we will have $|f(x') - f(x)| < \epsilon$. This proves continuity of f at $x \in \mathbf{R} - \mathbf{Q}$.

HOMEWORK 8

17.6. Prove that every rational function is continuous.

`Solution.` A rational function is obtained from constants and the identity function $y = x$ by the operations of multiplication, addition, and division. Since the identity and constant functions are obviously continuous, the result follows from the theorems about continuity of sums, products, and ratios of continuous functions.

18.2. Where does the proof of Theorem 18.1 (p. 126 of the textbook) break down if the domain of the function is an open (rather than closed) interval (a, b)?

`Solution.` The limit x_0 (or y_0) of the subsequence $x_{n_k} \in (a, b)$ (resp. $y_{n_k} \in (a, b)$) may be an endpoint a or b of the interval and thus lie outside the domain of the function.

18.4. Let $S \subset \mathbf{R}$ be not closed. Show that there exists an unbounded continuous function on S.

`Solution.` Let $x_0 \notin S$ be such that there exists a sequence (x_n) in S which converges to x_0 (such a number x_0 exists since S is not closed). Then the function $f(x) = |x - x_0|$ (i.e. the distance to x_0) is continuous and strictly positive on S. Thus $1/f$ is well-defined and continuous on S. It is unbounded since $\lim 1/|x_n - x_0| = \infty$.

18.6. Prove that $x = \cos x$ for some x on $(0, \pi/2)$.

`Solution.` Indeed, $f(x) := x - \cos x$ is continuous on $[0, \pi/2]$, negative at $x = 0$, and positive at $x = \pi/2$. Therefore, by the Intermediate Value Theorem, there exists an $x \in (0, \pi/2)$ such that $f(x) = 0$.

18.10. Suppose that f is continuous on $[0, 2]$, and $f(0) = f(2)$. Prove that there exist $x, y \in [0, 2]$ such that $|x - y| = 1$, and $f(x) = f(y)$.

`Solution.` Put $g(x) = f(x+1) - f(x)$. Then g is defined and continuous on $[0,1]$, $g(0) = f(1) - f(0) = f(1) - f(2) = -g(1)$. By the intermediate value theorem, there exists $x \in [0,1]$ such that $g(x) = 0$, i.e. $f(x+1) = f(x)$.

HOMEWORK 9

19.2b. Verify uniform continuity of $f(x) = x^2$ on $[0,3]$.

`Solution.` For a given positive ϵ, take $\delta = \epsilon/6$. then for any $x, y \in [0,3]$ such that $|x-y| < \delta$, we have $|f(x) - f(y)| = |x^2 - y^2| = |x+y| \cdot |x-y| < (3+3)\delta = \epsilon$.

19.4a. Prove that a function uniformly continuous on a bounded set is bounded.

`Solution.` Suppose that f is uniformly continuous but unbounded. Then there exists a sequence (x_n) in the domain of f such that $|f(x_n)| \geq n$. Since the domain is bounded, the sequence contains a convergent subsequence (x_{n_k}) (by the Bolzano-Weierstrass theorem). A convergent subsequence is Cauchy, and therefore the sequence of values $f(x_{n_k})$ is Cauchy by the property of uniformly continuous functions. But the sequence $|f(x_{n_k})| \geq n_k$ is unbounded — contradiction.

19.6a. Show that $f(x) = \sqrt{x}$ is uniformly continuous on $(0,1]$ although f' is unbounded.

`Solution.` The derivative $f'(x) = 1/2\sqrt{x}$ tends to ∞ as $x \to 0$, and is therefore unbounded. However, since f is continuous on the closed interval $[0,1]$, it is uniformly continuous on $[0,1]$, and therefore on the subset $(0,1]$ as well.

20.18. Show that $\lim_{x\to 0}(\sqrt{1+3x^3} - 1)/x^2$ exists, and find its value.

`Solution.` We have:

$$\frac{\sqrt{1+3x^2}-1}{x^2} = \frac{(1-3x^2)-1}{(\sqrt{1+3x^2}+1)x^2} = \frac{-3}{\sqrt{1+3x^2}+1},$$

which is a composition of continuous functions (the polynomial $x \mapsto 1+3x^2$, $y \mapsto \sqrt{y}$, addition of 1, inversion, multiplication by 3), well-defined near $x = 0$. Thus, the function has a limit as $x \to 0$, equal to the value of the function at $x = 0$, i.e. to $-3/2$.

19.10. Show that the function g such that $g(x) = x^2 \sin 1/x$ for $x \neq 0$, and $g(0) = 0$, is continuous on $\mathbf{R}$, and find out if it is uniformly continuous.

`Solution.` The limit

$$\lim_{x\to 0} \frac{g(x)}{x} = \lim_{x\to 0} x \sin\frac{1}{x} = 0,$$

i.e. g is differentiable at $x = 0$ (and $g'(0) = 0$). At $x \neq 0$,

$$g'(x) = 2x\sin\frac{1}{x} + x^2\cos\frac{1}{x}\cdot(-\frac{1}{x^2}) = 2x\sin\frac{1}{x} - \cos\frac{1}{x},$$

which is bounded. Indeed, since $|\cos y)| \neq 1$, and $|\sin y| \leq |y|$ for any y, and hence for $y = 1/x$, we have:

$$|2x\sin\frac{1}{x} - \cos\frac{1}{x}| = |\frac{2}{y}\sin y - \cos y| \leq 2 + 1 = 3.$$

Thus, g' is defined and bounded on $\mathbf{R}$, and therefore g is uniformly continuous.

HOMEWORK 10

28.15. Prove Leibniz' rule: $(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$.

`Solution:` Induction on n. For $n = 1$, Leibniz' rule turns into the "product rule" $9fg)' = f'g + fg'$.

Suppose that Leibniz' rule holds true for a given $n = m$. Then

$$\begin{aligned}(fg)^{(m+1)} = (f'g + fg')^{(m)} &= \sum_{k=0}^{m} \binom{m}{k} f^{(k+1)} g^{(m-k)} + \sum_{k=0}^{m} \binom{m}{k} f^{(k)} g^{(m+1-k)} \\ &= \sum_{k=0}^{m+1} \left[\binom{m}{k-1} + \binom{m}{k}\right] f^{(k)} g^{(m+1-k)} \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} f^{(k)} g^{(m+1-k)},\end{aligned}$$

where the last equality is due to the defining property of Pascal's triangle: $\binom{m+1}{k} = \binom{m}{k-1} + \binom{m}{k}$.

28.4c. For $g(x) = x^2 \sin 1/x$ at $x \neq 0$, and $g(0) = 0$, show that g' is not continuous at $x = 0$.

`Solution.` According to the solution to **19.10**, g is differentiable everywhere, $g(0) = 0$, and $g'(x) = x\sin 1/x - \cos 1/x$ at $x \neq 0$. The summand $x \sin 1/x$ tends to 0 as $x \to 0$, and $\cos 1/x$ has no limit at $x = 0$. Thus the sum $g'(x)$ has no limit at $x \neq 0$, and hence g' is discontinuous at $x = 0$.

29.10. For $f(x) = x/2 + x^2 \sin 1/x$ at $x \neq 0$, and $f(0) = 0$, show that $f'(0) > 0$, but f is not increasing on any interval containing 0, and compare this result with the theorem 29.7 (i) saying that a function is increasing on a given interval if it has positive derivative on it.

`Solution.` We have:

$$f'(0) = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0}\left[\frac{1}{2} + x\sin\frac{1}{x}\right] = \frac{1}{2} + 0 > 0.$$

On the other hand, at $x \neq 0$, we have

$$f'(x) = 2x\sin\frac{1}{x} + \frac{1}{2} - \cos\frac{1}{x}.$$

While the first term tends to 0 as $x \to 0$, the last term oscillates between -1 and 1 infinitely many times in any neighborhood of $x = 0$. Since $1/2 - 1 < 0$, in any neighborhood of $x = 0$, the derivative f' stays negative on some intervals. By the result of Theorem 29.7, the function f is decreasing on these intervals. Thus f is not increasing in any neighborhood of $x = 0$.

If f' were continuous at $x = 0$, then it would remain positive in some neighborhood of $x = 0$ (since $f'(0) = 1/2 > 0$, and thus f would have been increasing in that neighborhood. Thus this counter-example is due to discontinuity of f' at $x = 0$.

29.15. Show that for $r = m/n$, the derivative of the function $f(x) = x^r$ (in the appropriate domain depending on m and n) is equal to rx^{r-1}.

`Solution.` Since $(x^m)' = mx^{m-1}$ for $m \geq 0$ (induction on m based on the product rule), $1/x)' = -1/x^2$ (the theorem about derivatives of reciprocal functions), and $(x^{1/n})' = x^{\frac{1}{n}-1}$ (derivatives of inverse functions), the derivative of the function $x^{m/n} := (x^m)^{\frac{1}{n}}$ can be computed by the chain rule:

$$\frac{d}{dx}(x^m)^{\frac{1}{n}} = \frac{y^{\frac{1}{n}-1}}{n}|_{y=x^m} \cdot mx^{m-1} = \frac{m}{n}x^{m(\frac{1}{n}-1)}x^{m-1} = \frac{m}{n}x^{\frac{m}{n}-1}.$$

29.18. Prove that a sequence (s_n), defined recursively by the rule $s_{n+1} := f(s_n)$ and a choice of s_0, converges provided that f is a function differentiable on $\mathbf{R}$ with the derivative bounded in the absolute value by a number $a < 1$.

`Solution.` By the mean value theorem, for each $n > 0$, we have:

$$|s_{n+1} - s_n| = |f(s_n) - f(s_{n-1})| = |f'(y)(s_n - s_{n-1}| \leq a|s_n - s_{n-1}|.$$

By induction, this implies: $|s_{n+1} - s_n| \leq a^n |s_1 - s_0|$ for all $n > 0$. Furthermore, for any $m \geq n > 0$, we have:

$$|s_{m+1} - s_n| \leq \sum_{k=n}^{m} |s_{k+1} - s_k| \leq (s_1 - s_0) \sum_{k=n}^{m} a^k.$$

Since the geometric series $\sum a^k$ with $a < 1$ converges, its partial sums $\sum_{k=0}^{n} a^k$ form a Cauchy sequence. The previous estimate implies therefore that (s_n) is a Cauchy sequence, and hence converges by the completeness property of $\mathbf{R}$.

HOMEWORK 11

We have $1/\tan(y + \pi/2) = 1/\cot(-y) = -\tan y = -y + o(|y|)$, and $1/\tan(3y + 3\pi/2) = 1/\cot(-3y) = -\tan 3y = -3y + o(|y|)$. Therefore

(a) $$\lim_{x \to \pi/2} \frac{\tan 3x}{\tan x} = \lim_{y \to 0} \frac{-\tan y}{-\tan 3y} = \frac{y + o(|y|)}{3y + (|y|)} = \frac{1}{3}.$$

Furthermore, we have $\tan x = x + x^3/3 + o(|x|^3)$, $\sin x = x - x^3/6 + o(|x|^3)$. Therefore

(b)
$$\lim_{x \to 0} \frac{3\tan 4x - 12\tan x}{3\sin 4x - 12\sin x} = \lim_{x \to 0} \frac{12x + 64x^3 - 12x - 4x^3 + o(|x^3|)}{12x - 32x^3 - 12x + 2x^3 + o(|x|^3)} = -2.$$

Since $\lim_{x\to 0} \sin bx/\sin ax = b/a$, we have by l'Hospital's rule:

(c) $$\lim_{x \to 0^+} \frac{\ln(\sin ax)}{\ln(\sin bx)} = \lim_{x \to 0^+} \frac{a\cos ax/\sin ax}{b\cos bx/\sin bx} = \frac{a}{b}\frac{b}{a} = 1.$$

Next, $\lim_{x\to 0} x\ln x = 0$ (by l'Hospital's rule, or just because $x = e^{-t}$ decreases much fastes as $t \to \infty$ than $\ln x = -t$ grows in the absolute value). Therefore $x^x = \exp(x \ln x)$ tends to 1 as $x \to 0^+$. Hence $x^x \ln x = \ln x^{x^x}$ tends to $-\infty$ as $x \to 0^+$, and thus x^{x^x} tends to 0. We conclude

(d) $$\lim_{x \to 0^+} (x^{x^x} - 1) = 0 - 1 = -1.$$

By l'Hospital's rule,

$$\lim_{y \to \infty} \frac{y^{2n}}{e^{y^2}} = \frac{ny^{n-2}}{e^{y^2}},$$

if the latter limit exists. Since e^{-y^2} tensd to 0 as $y \to \infty$, we conclude by induction on $n \geq 0$ that the limits exist and are equal to 0. Taking $y = 1/x$ we conclude that

(e) $$\lim_{x \to 0} \frac{e^{-1/x^2}}{x^{100}} = 0.$$