

Math 123. Take Home Final

Rules: Solve the following problems and submit your written solutions at 11 a.m. on Tuesday, December 9, 2008. You may bring your solutions to the class (or submit them in any other way, e.g. by email to givental@math.berkeley.edu).

While solving the problems, you may not consult other people or any sources (including Internet), with the exception of the textbook for this course as well as one textbook of your choice in each: Linear Algebra, Multivariable Calculus, and Introductory Analysis. Please provide references to the books you actually use.

Breaking the above rule constitutes a serious offense against the *honor code* thereby imposed on you, and may result in severe penalties. By submitting your final exam you acknowledge that you followed the rules.

1. Let v be a vector field on the (x, y) -plane defined as the gradient of the function

$$f(x, y) = \frac{x^2y}{2} + \frac{y^3}{3} - 2axy - b^2y,$$

where a and b are non-zero parameters.

(a) Find singular points of the vector field v , and for each of them, determine if it is (Lyapunov and asymptotically) stable or unstable. Specify corresponding ranges of the parameters if the conclusion depends on their values.

(b) For the linear approximation to the vector field near each singular point, determine the type of the singular point (i.e. decide if it is a stable or unstable node, a saddle, a stable or unstable focus, etc.), and sketch the corresponding phase portrait.

2. A particle of mass $m = 1$ is moving without friction in the plane with coordinates (x_1, x_2) , subject to the conservative force field with the potential energy function $U = x_1^2 + x_1x_2 + x_2^2$.

(a) In the phase space of this system, find two linearly independent conservation laws (in another terminology, *first integrals*) which are quadratic forms (i.e. homogeneous quadratic functions of the phase variables $x_1, x_2, \dot{x}_1, \dot{x}_2$).

(b) Let F and G denote the quadratic conservation laws found in part (a). Is it true that every continuous¹ conservation law of this system is a (continuous) function of F and G , i.e. can be represented as a composite function $H(F(x, \dot{x}), G(x, \dot{x}))$? If “no,” give a counter-example, if “yes,” provide a proof.

¹but not necessarily quadratic, of course