Math H104. Take Home Final

`Rules:` Solve the following problems and submit your written solutions at 3 : 30 p.m. on Tuesday, December 9, 2008. You may bring your solutions to the class (or submit them in any other way, e.g. email to givental@math.berkeley.edu).

While solving the problems, you may not consult other people or any sources (including Internet), with the exception of the textbook for this course.

Breaking the above rule constitutes a serious offense against the *honor code* thereby imposed on you, and may result in severe penalties. By submitting your final exam you acknowledge that you followed the rules.

1. Let $f : [0, \infty) \to \mathbf{R}$ be a continuous compactly supported function (i.e. $f(x) = 0$ for all sufficiently large x). Define the *L-transform*[1] of f as

$$\hat{f}(z) = \int_0^\infty e^{-x} f(xz) dx.$$

(a) Prove that $\hat{f}$ is infinitely differentiable at all $z > 0$.

(b) Suppose that f is infinitely differentiable. Prove that $\hat{f}$ is infinitely differentiable at $z = 0$ as well, and that

$$\hat{f}^{(n)}(0) = n! f^{(n)}(0) \text{ for all } n = 0, 1, 2, \ldots.$$

(c) Prove that if the Taylor series of f at $x = 0$ has a finite convergence radius, then the Taylor series of $\hat{f}$ has the zero convergence radius.

(d) Give an example of an infinitely differentiable function whose Taylor series has the zero convergence radius.

2. Let $P_1, P_2, P_3, \ldots,$ be a sequence of non-zero polynomials on $\mathbf{R}^n$.

(a) Prove that the subset

$$\{x \in \mathbf{R}^n | \forall i,\ P_i(x) \neq 0\}$$

is dense, i.e. every neighborhood of every point in $\mathbf{R}^n$ contains a point where none of the polynomials vanish.

(b) For $n = 1$, give an example showing that the statement fails, when P_i are not required to be polynomials, but can be arbitrary infinitely differentiable functions.

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[1] It is closely related to Laplace's transform.

3. Given $M \geq 0$ and $L \geq 0$, denote by $E_{M,L}$ the set of all differentiable 2π-periodic functions $f : \mathbf{R} \to \mathbf{R}$ satisfying

$$|f'(x)| \leq M \text{ for all } x, \text{ and } \quad \left| \int_0^{2\pi} f(x)dx \right| \leq L.$$

(a) Prove that every sequence $\{f_n\}$ of functions from $E_{M,L}$ contains a uniformly convergent subsequence.

(b) Prove that (the supremum over all functions from $E_{M,L}$)

$$S_{M,L} := \sup_{f \in E_{M,L}} \quad \int_0^{2\pi} |f(x)|dx$$

is finite (i.e. $S < \infty$.

(c) In the special case $M = 1, L = 0$, find $S_{1,0}$.