SOLUTIONS

(a) $$\sum_{n\geq 1} \frac{1}{\sqrt{n(n+1)}}$$

diverges by the comparison test: $1/\sqrt{n(n+1)} > 1/(n+1)$ for all $n \in \mathbf{N}$.

(b) $$\sum_{n\geq 1} \frac{(n!)^2}{(2n)!}$$

converges by the ratio test: $\lim |a_n/a_{n-1}| = \lim n^2/2n(2n-1) = 1/4 < 1$.

(c) $$\sum_{n\geq 1} \frac{n!}{n^n}$$

converges by the ratio test: $\lim |a_{n+1}/a_n| = \lim (1+1/n)^{-n} = e^{-1} < 1$.

(d) $$\sum_{n\geq 1} \frac{(n!)^2}{2^{n^2}}$$

converges by the ratio test: $\lim |a_n/a_{n-1}| = \lim n^2/2^{2n-1} = 0 < 1$.

(e) $$\frac{1000}{1} + \frac{1000\cdot 1001}{1\cdot 3} + \frac{1000\cdot 1001\cdot 1002}{1\cdot 3\cdot 5} + \cdots$$

converges by the ratio test: $\lim |a_n/a_{n-1}| = \lim (1000+n)/(2n+1) = 1/2 < 1$.

(f) $$\sum_{n\geq 1} (2^{1/2} - 2^{1/3})(2^{1/2} - 2^{1/5}) \cdots (2^{1/2} - 2^{1/(2n+1)})$$

converges by the ratio test: $\lim |a_n/a_{n-1}| = \lim (2^{1/2} - 2^{1/2n+1}) = \sqrt{2} - 1 < 1$.

(g) $$\sum_{n\geq 1} \frac{n^2}{(2+\frac{1}{n})^n}$$

converges by the root test: $\lim |a_n|^{1/n} = \lim n^{2/n}/(2+1/n) = 1/2 < 1$.

(h) $$\sum_{n\geq 1} \frac{n^{n+\frac{1}{n}}}{(n+\frac{1}{n})^n}$$

diverges since $\lim a_n^n/n = \lim (1+1/n^2)^{-n^2} = 1/e$, i.e. for large n, $a_n > 1$ (since otherwise $\lim a_n^n/n = 0$).

(i) $$\sum_{n=1}^{\infty} a_n \quad \text{where } a_n := \begin{cases} 1/n & \text{if } n = m^2 \\ 1/n^2 & \text{if } n \neq m^2. \end{cases}$$

Partial sums of this series form an increasing sequence which is bounded by the sum of $\sum_{n=1}^{\infty} 1/n^2$ with $\sum_{m=1}^{\infty} 1/m^2$. Thus the series converges.

(j) $\sqrt{2} + \sqrt{2-\sqrt{2}} + \sqrt{2-\sqrt{2+\sqrt{2}}} + \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}} + \cdots$

converges by the ratio test. Indeed, $a_n = \sqrt{2-b_n}$, where $b_0 = 0$, $b_n = \sqrt{2+b_{n-1}} > 0$ for $n > 0$. Then

$$a_n = \frac{\sqrt{2-b_n}\sqrt{2+b_n}}{\sqrt{2+b_n}} = \frac{\sqrt{4-b_n^2}}{\sqrt{2+b_n}} = \frac{\sqrt{2-b_{n-1}}}{\sqrt{2+b_n}} = \frac{a_{n-1}}{\sqrt{2+b_n}}.$$

Thus $\limsup |a_n/a_{n-1}| \le 1/\sqrt{2} < 1$.